Homogenization of elasto–plastic plate equations with vanishing hardening

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Outline

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1. Linearized elasto-plasticity and problem setup
2. Dimension reduction of heterogeneous elasto-plastic plate model with hardening
3. Simultaneous homogenization and vanishing of hardening

Key objects in linearized elasto-plasticity

- displacement vector:

  $u : [0,T] \times \Omega \to \mathbb{R}^n$

- linearized strain tensor:

  $\mathbf{E}u := \operatorname{sym} \nabla u = \frac{1}{2}(\nabla u + \nabla u^T)$

  elastic strain tensor:

  $e : [0,T] \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$

  plastic strain tensor:

  $p : [0,T] \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{dev}} := \left\{A \in \mathbb{M}^{n\times n}_{\mathrm{sym}} \,:\, \operatorname{tr} A = 0\right\}$

- elasticity domain:

  $K \subset \mathbb{M}^{n\times n}_{\mathrm{dev}}$ convex, compact neighbourhood of 0

- stress tensors:

  $\sigma : [0,T] \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$

  $\chi_{\mathrm{kin}} : [0,T] \times \Omega \to \mathbb{M}^{n\times n}_{\mathrm{dev}}$

  $\chi_{\mathrm{iso}} : [0,T] \times \Omega \to \mathbb{R}$

- internal isotropic hardening variable:

  $\alpha : [0,T] \times \Omega \to \mathbb{R}$

Perfect plasticity

Find $(u,\, e,\, p)$ such that $\forall t \in [0,T]$:

(c1) $\mathbf{E}u = e + p$ for $x \in \Omega$

$u = w$ for $x \in \Gamma_D$

(c2) $\sigma = \mathbb{C}e$

(c3) $-\mathrm{div}_x\, \sigma = f$ for $x \in \Omega$

$\sigma \cdot \nu(x) = g$ for $x \in \partial\Omega \setminus \Gamma_D$

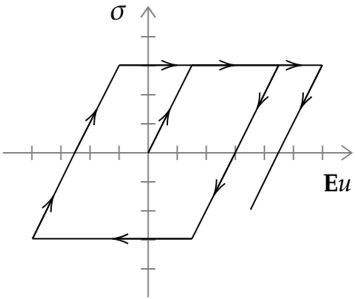
(c4) $\sigma_{\mathrm{dev}} \in K$

(c5) If $\sigma_{\mathrm{dev}} \in \mathrm{int}K$, then $\dot{p} = 0$.

If $\sigma_{\mathrm{dev}} \in \partial K$, then $\dot{p} \perp \partial K$ at σ_{dev}.

$$\Updownarrow$$

$\sigma_{\mathrm{dev}} \in \partial\mathbf{R}(\dot{p})$, where $\mathbf{R}(p) := \sup\limits_{\varsigma \in K} \varsigma : p$.



$$\rightsquigarrow \quad \mathcal{E}(t,u,e,p) := \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx - \langle \ell(t), u\rangle$$

$$\mathcal{R}(p) := \int_\Omega \mathbf{R}(p)\,dx$$

The natural setting for perfect plasticity

We use the space

$$BD(\Omega) = \Big\{ u \in L^1(\Omega;\mathbb{R}^n) : \mathbf{E}u \in \mathcal{M}_b(\Omega;\mathbb{M}^{n\times n}_{\mathrm{sym}}) \Big\}.$$

Since $p \in \mathcal{M}_b(\Omega \cup \Gamma_D;\mathbb{M}^{n\times n}_{\mathrm{dev}})$, we interpret

$$\mathcal{R}(p) = \int_{\Omega\cup\Gamma_D} \mathbf{R}\left(\frac{dp}{d|p|}\right)\, d|p|.$$

Moreover, the boundary condition is relaxed by requiring that

$$p = (w-u)\odot \nu\mathcal{H}^{n-1} \;\text{ on } \Gamma_D.$$

Plasticity with kinematic hardening

Find $(u,\, e,\, p)$ such that $\forall t \in [0,T]$:

(c1) $\mathbf{E}u = e + p$ for $x \in \Omega$

$u = w$ for $x \in \Gamma_D$

(c2) $\sigma = \mathbb{C}e$

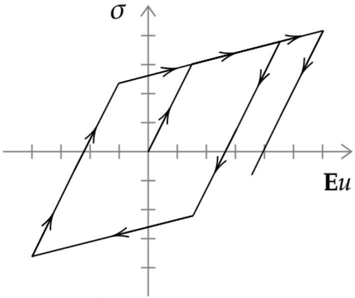
$\chi_{\mathrm{kin}} = -\mathbb{H}_{\mathrm{kin}} p$

(c3) $-\mathrm{div}_x\, \sigma = f$ for $x \in \Omega$

$\sigma \cdot \nu(x) = g$ for $x \in \partial\Omega \setminus \Gamma_D$

(c4) $\sigma_{\mathrm{dev}} + \chi_{\mathrm{kin}} \in K$

(c5) $\sigma_{\mathrm{dev}} + \chi_{\mathrm{kin}} \in \partial\mathbf{R}(\dot{p})$



$$\rightsquigarrow \quad \mathcal{E}(t,u,e,p) := \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx + \frac{1}{2}\int_\Omega \mathbb{H}_{\mathrm{kin}} p : p\,dx - \langle \ell(t), u\rangle$$

$$\mathcal{R}(p) := \int_\Omega \mathbf{R}(p)\,dx$$

Plasticity with isotropic hardening

Find $(u,\, e,\, p,\, \alpha)$ such that $\forall t \in [0,T]$:

(c1) $\mathbf{E}u = e + p$ for $x \in \Omega$

$u = w$ for $x \in \Gamma_D$

(c2) $\sigma = \mathbb{C}e$

$\chi_{\text{iso}} = -H_{\text{iso}}\alpha$

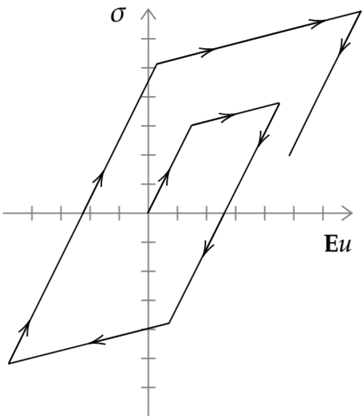
(c3) $-\text{div}_x\, \sigma = f$ for $x \in \Omega$

$\sigma \cdot \nu(x) = g$ for $x \in \partial\Omega \setminus \Gamma_D$

(c4) $\sigma_{\text{dev}} \in (1 - \chi_{\text{iso}})\, K$

(c5) $(\sigma_{\text{dev}}, \chi_{\text{iso}}) \in \partial\widehat{\mathbf{R}}(\dot{p}, \dot{\alpha})$, where

$$\widehat{\mathbf{R}}(p,\alpha) := \begin{cases} \mathbf{R}(p) & \text{if } \mathbf{R}(p) \le \alpha, \\ +\infty & \text{else.} \end{cases}$$



$$\rightsquigarrow \quad \mathcal{E}(t,u,e,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx + \frac{1}{2}\int_\Omega H_{\text{iso}}\alpha\cdot\alpha\,dx - \langle \ell(t), u\rangle$$

$$\mathcal{R}(p,\alpha) := \int_\Omega \widehat{\mathbf{R}}(p,\alpha)\,dx$$

The variational formulation for rate-independent processes

A function $q : [0,T] \to \boldsymbol{X}$ is an energetic solution of a rate-independent process $(\boldsymbol{X}, \mathcal{E}, \mathcal{R})$ if for every $t \in [0,T]$ it satisfies:

(1) Global stability:

$$\mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}(\tilde{q} - q(t)),$$

for every comparable state $\tilde{q} \in \boldsymbol{X}$

(2) Energy balance:

$$\underbrace{\mathcal{E}(t, q(t))}_{\text{present energy}} + \underbrace{\mathcal{V}_{\mathcal{R}}(q; 0, t)}_{\text{dissipated energy}} = \underbrace{\mathcal{E}(0, q(0))}_{\text{initial energy}} + \underbrace{\int_0^t \partial_t \mathcal{E}(s, q(s))\, ds}_{\text{work of external forces}}$$

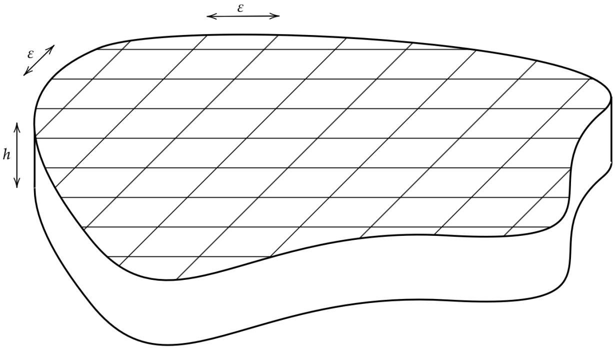
where $\mathcal{V}_{\mathcal{R}}(q; a, b) := \sup \left\{ \sum_{i=1}^{n-1} \mathcal{R}\big(q(t_{i+1}) - q(t_i)\big) : \text{ all partitions of } [a,b] \right\}$

A heterogeneous perfectly-plastic thin plate

Let $\omega \subset \mathbb{R}^2$ be a bounded, connected, and open set with a C^2 boundary. We define

$$\Omega^h := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \Gamma_D^h := \gamma_D \times \left(-\frac{h}{2}, \frac{h}{2}\right).$$

We assume for simplicity that $\gamma_D = \partial\omega$.



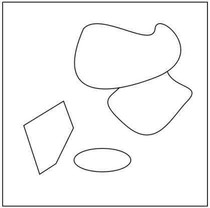
A heterogeneous perfectly-plastic thin plate

The torus $\mathcal{Y} := \mathbb{R}^2/\mathbb{Z}^2$ is made up of finitely many phases $\mathcal{Y}_i$ together with their interfaces $\Gamma_{ij} = \partial\mathcal{Y}_i \cap \partial\mathcal{Y}_j$. We will write

$$\Gamma := \bigcup_{i \neq j} \Gamma_{ij}.$$

We assume that there exists a compact set $S \subset \Gamma$ with $\mathcal{H}^1(S) = 0$ such that

$$\Gamma \setminus S \quad \text{is a } C^2\text{-hypersurface.}$$



We assume there exist convex compact sets $K_i \subset \mathbb{M}^{3\times 3}_{\mathrm{dev}}$ for each phase $\mathcal{Y}_i$.
Finally, we define

$$K(y) := K_i, \quad \text{for } y \in \mathcal{Y}_i.$$

Short literature overview

Existence results for perfect plasticity in the quasistatic setting were originally provided in

- P. Suquet (1982)

and subsequently reformulated within the theory of rate-independent processes in

- G. Dal Maso, A. DeSimone and M.G. Mora (2006)
- G.A. Francfort and A. Giacomini (2012)

Evolution of perfectly-plastic models can be obtained as the limit from the classical elasto-plastic model with hardening, if the hardening approaches zero:

- S. Bartels, A. Mielke, T. Roubíček (2012)

Short literature overview

Dimension reduction in plasticity:

- E. Davoli and M.G. Mora (2013, 2015)
- M. Liero and A. Mielke (2011), M. Liero and T. Roche (2012)

Homogenization in plasticity:

- G.A. Francfort and A. Giacomini (2014)
- A. Mielke and A.M. Timofte (2007)
- B. Schweizer and M. Veneroni (2015),
  M. Heida and B. Schweizer (2016, 2018)

Simultaneous homogenization and dimension reduction in plasticity:

- M.B., E. Davoli and I. Velčić (2024)

$$\lim_{h \to 0} \frac{h}{\varepsilon_h} =: \gamma \in [0, +\infty]$$

Pointwise definition of the dissipation potential

Pointwise $\mathbf{R} : \mathcal{Y} \times \mathbb{M}^{3\times 3}_{\mathrm{dev}} \to [0, +\infty]$ can be defined as follows:

(a) For every $y \in \mathcal{Y}_i$, we take

$$\mathbf{R}(y, M) := \mathbf{R}_i(M) = \sup_{\varsigma \in K_i} \varsigma : M.$$

(b) If $y \in \Gamma_{ij} \setminus S$, then:

(i) For $\gamma \in (0, +\infty)$,

$$\mathbf{R}(y, M) := \begin{cases} \mathbf{R}_{ij}(a, \nu(y)) & \text{if } M = a \odot \nu(y), \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathbf{R}_{ij}(a, \nu) := \inf \Big\{ \mathbf{R}_i(a_i \odot \nu) + \mathbf{R}_j(-a_j \odot \nu) : a = a_i - a_j,\, a_i \perp \nu,\, a_j \perp \nu \Big\}.$$

(ii) For $\gamma = 0$ or $\gamma = +\infty$,

$$\mathbf{R}(y, M) = \min_{i,j} \Big\{ \mathbf{R}_i(M),\, \mathbf{R}_j(M) \Big\}.$$

only if ordering between the phases is assumed on the interface: at points where exactly two phases $\mathcal{Y}_i$ and $\mathcal{Y}_j$ meet we have that either $K_i \subset K_j$ or $K_j \subset K_i$

Outline

Rescaled problem

Let $I = \left(-\frac{1}{2}, \frac{1}{2}\right)$ and set

$$\Omega := \omega \times I, \quad \Gamma_D := \gamma_D \times I.$$

We consider the change of variables $\psi_h : \Omega \to \Omega^h$ defined as

$$\psi_h(x', x_3) := (x', hx_3)$$

and the linear operator $\Lambda_h : \mathbb{M}^{3\times 3}_{\text{sym}} \to \mathbb{M}^{3\times 3}_{\text{sym}}$ given by

$$\Lambda_h M := \left[\begin{array}{c|c} M_{\alpha\beta} & \frac{1}{h} M_{\alpha 3} \\ \hline \frac{1}{h} M_{3\beta} & \frac{1}{h^2} M_{33} \end{array}\right].$$

For $(\upsilon, \eta, \pi, \beta)$ kinematically admissible in Ω^h we set

$$\begin{aligned} u^h &:= (\upsilon_1,\, \upsilon_2,\, h\upsilon_3) \circ \psi_h, \\ e^h &:= \Lambda_h^{-1} \eta \circ \psi_h, \quad p^h := \Lambda_h^{-1} \pi \circ \psi_h, \\ \alpha^h &:= \beta \circ \psi_h. \end{aligned}$$

Admissible configurations and energies ($h > 0$)

The class $\mathcal{A}_h^{\mathrm{hard}}(w)$ of h-admissible displacements and strains is given by all tuples

$$(u^h, e^h, p^h, \alpha^h) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{M}^{3\times3}_{\mathrm{sym}}) \times L^2(\Omega)$$

satisfying:

$$\begin{aligned}
&\mathbf{E}u^h = e^h + p^h \quad \text{in } \Omega,\\
&u^h = w \quad \mathcal{H}^2\text{-a.e. on } \Gamma_D,\\
&\mathbf{R}(x', \Lambda_h p^h(x)) \leq \alpha^h(x) \quad \text{a.e. in } \Omega,\\
&p^h_{11} + p^h_{22} + \frac{1}{h^2} p^h_{33} = 0 \quad \text{in } \Omega
\end{aligned}$$

For $(u^h, e^h, p^h, \alpha^h) \in \mathcal{A}_h^{\mathrm{hard}}(w)$ we define the associated energy functionals as

$$\begin{aligned}
\mathcal{Q}_h^{\mathrm{el}}(e^h) &:= \frac{1}{2}\int_\Omega \mathbb{C}(x')\Lambda_h e^h : \Lambda_h e^h \, dx,\\
\mathcal{Q}_h^{\mathrm{hard}}(p^h, \alpha^h) &:= \frac{1}{2}\int_\Omega \mathbb{H}_{\mathrm{kin}}(x')\Lambda_h p^h : \Lambda_h p^h \, dx + \frac{1}{2}\int_\Omega H_{\mathrm{iso}}(x')\alpha^h \cdot \alpha^h \, dx,\\
\mathcal{R}_h(p^h, \alpha^h) &:= \int_\Omega \widehat{\mathbf{R}}(x', \Lambda_h p^h, \alpha^h) \, dx.
\end{aligned}$$

Γ-convergence result 1: compactness result

Proposition

Let $\{u^h\}_{h>0} \subset H^1(\Omega;\mathbb{R}^3)$ and let there exists a constant $C > 0$ for which

$$\|u^h\|_{L^2(\Omega;\mathbb{R}^3)} + \|\Lambda_h \mathbf{E} u^h\|_{L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}})} \leq C, \qquad \text{for every } h > 0.$$

Then there exist $\bar{u} = (\bar{u}_1, \bar{u}_2) \in H^1(\omega;\mathbb{R}^2)$ and $u_3 \in H^2(\omega)$ which satisfy

$$u^h_\alpha \longrightarrow \bar{u}_\alpha - x_3 \partial_{x_\alpha} u_3 \quad \text{strongly in } L^2(\Omega), \quad \alpha \in \{1,2\},$$
$$u^h_3 \longrightarrow u_3 \quad \text{strongly in } L^2(\Omega),$$
$$\mathbf{E} u^h \rightharpoonup \begin{pmatrix} \mathbf{E}_{x'} \bar{u} - x_3 \nabla^2_{x'} u_3 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}).$$

The reduced elasticity tensor

For a fixed $x' \in \omega$, let $\mathbb{M}_{x'} : \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ be the operator given by

$$\mathbb{M}_{x'}\xi := \begin{pmatrix} & \xi & \lambda_1^{x'}(\xi) \\ & & \lambda_2^{x'}(\xi) \\ \lambda_1^{x'}(\xi) & \lambda_2^{x'}(\xi) & \lambda_3^{x'}(\xi) \end{pmatrix} \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}},$$

where for every $\xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ the tuple $(\lambda_1^{x'}(\xi), \lambda_2^{x'}(\xi), \lambda_3^{x'}(\xi))$ is the unique solution to the minimum problem

$$\min_{\lambda_i^{x'} \in \mathbb{R}} \mathbb{C}(x') \begin{pmatrix} & \xi & \lambda_1^{x'} \\ & & \lambda_2^{x'} \\ \lambda_1^{x'} & \lambda_2^{x'} & \lambda_3^{x'} \end{pmatrix} : \begin{pmatrix} & \xi & \lambda_1^{x'} \\ & & \lambda_2^{x'} \\ \lambda_1^{x'} & \lambda_2^{x'} & \lambda_3^{x'} \end{pmatrix}.$$

We then define the reduced elasticity tensor $\mathbb{C}^{\mathrm{red}} : \omega \times \mathbb{M}^{2\times 2}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$, given by

$$\mathbb{C}^{\mathrm{red}}(x')\xi := \mathbb{C}(x')\mathbb{M}_{x'}\xi \quad \text{for every } \xi \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}.$$

Admissible configurations and energies ($h = 0$)

Let $w \in H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)$. We take the limiting class $\mathcal{A}_0^{\rm hard}(w)$ of all tuples

$$(u, e, p, \alpha) \in (H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{\rm sym}) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\rm dev}) \times L^2(\Omega)$$

satisfying:

$$\begin{aligned} &\mathbf{E}_{x'}\bar{u} - x_3\nabla^2_{x'}u_3 = e + p^{1,2} \quad \text{in } \Omega, \\ &u = w \quad \mathcal{H}^2\text{-a.e. on } \Gamma_D, \\ &\mathbf{R}(x', p(x)) \leq \alpha(x) \quad \text{a.e. in } \Omega \end{aligned}$$

For $(u, e, p, \alpha) \in \mathcal{A}_0^{\rm hard}(w)$ we now define

$$\begin{aligned} \mathcal{Q}_0^{\rm el}(e) &:= \frac{1}{2}\int_\Omega \mathbb{C}^{\rm red}(x')e : e\,dx, \\ \mathcal{Q}_0^{\rm hard}(p,\alpha) &:= \frac{1}{2}\int_\Omega \mathbb{H}_{\rm kin}(x')p : p\,dx + \frac{1}{2}\int_\Omega H_{\rm iso}(x')\alpha\cdot\alpha\,dx, \\ \mathcal{R}_0(p,\alpha) &:= \int_\Omega \widehat{\mathbf{R}}(x', p, \alpha)\,dx. \end{aligned}$$

Γ-convergence result 2: liminf inequality

Proposition

Let $(u^h, e^h, p^h, \alpha^h) \in \mathcal{A}_h^{\mathrm{hard}}(w)$ be such that

$$\begin{aligned} u^h &\rightharpoonup u \quad \text{weakly in } H^1(\Omega;\mathbb{R}^3),\\ \Lambda_h e^h &\rightharpoonup \mathbb{M}_{x'} e \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{sym}}),\\ \Lambda_h p^h &\rightharpoonup p \quad \text{weakly in } L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{dev}}),\\ \alpha^h &\rightharpoonup \alpha \quad \text{weakly in } L^2(\Omega), \end{aligned}$$

with $(u, e, p, \alpha) \in \mathcal{A}_0^{\mathrm{hard}}(w)$. Then we get

$$\mathcal{Q}_0^{\mathrm{el}}(e) + \mathcal{Q}_0^{\mathrm{hard}}(p,\alpha) - \langle \ell(t), u \rangle \le \liminf_h \left(\mathcal{Q}_h^{\mathrm{el}}(e^h) + \mathcal{Q}_h^{\mathrm{hard}}(p^h,\alpha^h) - \langle \ell(t), u^h \rangle \right)$$

and

$$\mathcal{R}_0(p,\alpha) \le \liminf_h \mathcal{R}_h(p^h,\alpha^h).$$

Γ-convergence result 3: recovery sequence

Proposition

Let $(u, e, p, \alpha) \in \mathcal{A}_0^{\mathrm{hard}}(w)$. There exists a sequence $(u^h, e^h, p^h, \alpha^h) \in \mathcal{A}_h^{\mathrm{hard}}(w)$ such that

$$u^h \rightharpoonup u \quad \text{weakly in } H^1(\Omega; \mathbb{R}^3),$$
$$\Lambda_h e^h \to \mathbb{M}_{x'} e \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{sym}}),$$
$$\Lambda_h p^h \to p \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{3\times 3}_{\mathrm{dev}}),$$
$$\alpha^h \to \alpha \quad \text{strongly in } L^2(\Omega),$$
$$\mathcal{R}_h(p^h, \alpha^h) \to \mathcal{R}_0(p, \alpha).$$

The elasto-plastic plate model of heterogeneous thin plate

We can then justify the limit quasistatic evolution via evolutionary Γ–convergence.

If we use the following notation for zero-th and first order moments:

$$\bar{\varphi}(x') := \int_I \varphi(x', x_3)\, dx_3, \qquad \hat{\varphi}(x') := 12 \int_I x_3\, \varphi(x', x_3)\, dx_3,$$

then it can bee easily seen that the model implies the following system of equations in terms of the in-plane and out-of-plane displacements and plastic strains:

$$-\mathrm{div}_{x'}\Big(\mathbb{C}^{\mathrm{red}}(x')\left(\mathbf{E}_{x'}\bar{u} - \overline{p}^{1,2}\right)\Big) = f^{\mathrm{memb}}(t, x') \quad \text{on } \omega,$$

$$\frac{1}{12}\mathrm{div}_{x'}\mathrm{div}_{x'}\Big(\mathbb{C}^{\mathrm{red}}(x')\left(\nabla^2_{x'} u_3 + \widehat{p}^{1,2}\right)\Big) = f^{\mathrm{bend}}(t, x') \quad \text{on } \omega,$$

$$\begin{pmatrix} \left[\mathbb{C}^{\mathrm{red}}(x')(\mathbf{E}_{x'}\bar{u} - x_3\nabla^2_{x'}u_3 - p^{1,2})\right]_{\mathrm{dev}} - \mathbb{H}_{\mathrm{kin}}(x')p \\ -H_{\mathrm{iso}}(x')\alpha \end{pmatrix} \in \partial\widehat{\mathbf{R}}(x', \dot{p}, \dot{\alpha}) \quad \text{on } \Omega$$

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Admissible configurations and energies ($\varepsilon > 0$)

Introducing periodic microstructure for the limiting quasistatic evolution, we consider the class $\mathcal{A}_0^{\mathrm{hard},\varepsilon}(w)$ of all tuples

$$(u^\varepsilon, e^\varepsilon, p^\varepsilon, \alpha^\varepsilon) \in (H^1(\Omega;\mathbb{R}^3) \cap KL(\Omega)) \times L^2(\Omega;\mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times L^2(\Omega;\mathbb{M}^{3\times 3}_{\mathrm{dev}}) \times L^2(\Omega)$$

satisfying:

$$\begin{aligned}
&\mathbf{E}_{x'}\bar{u}^\varepsilon - x_3 \nabla^2_{x'} u_3^\varepsilon = e^\varepsilon + (p^\varepsilon)^{1,2} \quad \text{in } \Omega,\\
&u^\varepsilon = w \quad \mathcal{H}^2\text{-a.e. on } \Gamma_D,\\
&\mathbf{R}\left(\tfrac{x'}{\varepsilon}, p^\varepsilon(x)\right) \leq \alpha^\varepsilon(x)
\end{aligned}$$

For $(u^\varepsilon, e^\varepsilon, p^\varepsilon, \alpha^\varepsilon) \in \mathcal{A}_0^{\mathrm{hard},\varepsilon}(w)$ we now define

$$\begin{aligned}
\mathcal{Q}_0^{\mathrm{el},\varepsilon}(e^\varepsilon) &:= \frac{1}{2}\int_\Omega \mathbb{C}^{\mathrm{red}}\left(\tfrac{x'}{\varepsilon}\right) e^\varepsilon : e^\varepsilon \, dx,\\
\mathcal{Q}_0^{\mathrm{hard},\varepsilon}(p^\varepsilon, \alpha^\varepsilon) &:= \frac{1}{2}\int_\Omega \mathbb{H}_{\mathrm{kin}}\left(\tfrac{x'}{\varepsilon}\right) p^\varepsilon : p^\varepsilon \, dx + \frac{1}{2}\int_\Omega H_{\mathrm{iso}}\left(\tfrac{x'}{\varepsilon}\right) \alpha^\varepsilon \cdot \alpha^\varepsilon \, dx,\\
\mathcal{R}_0^\varepsilon(p^\varepsilon, \alpha^\varepsilon) &:= \int_\Omega \widehat{\mathbf{R}}\left(\tfrac{x'}{\varepsilon}, p^\varepsilon, \alpha^\varepsilon\right) dx.
\end{aligned}$$

Vanishing hardening

Let

$$\delta(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

A ε-limiting quasistatic evolution is a function

$$t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t), \alpha^\varepsilon(t)) \in \mathcal{A}_0^{\mathrm{hard},\varepsilon}(w(t))$$

which satisfies the following conditions for every $t \in [0,T]$:

(1) Global stability:

$$\begin{aligned} &\mathcal{Q}_0^{\mathrm{el},\varepsilon}(e^\varepsilon(t)) + \delta(\varepsilon)\, \mathcal{Q}_0^{\mathrm{hard},\varepsilon}(p^\varepsilon(t), \alpha^\varepsilon(t)) \\ &\le \mathcal{Q}_0^{\mathrm{el},\varepsilon}(\eta) + \delta(\varepsilon)\, \mathcal{Q}_0^{\mathrm{hard},\varepsilon}(\pi, \beta) + \mathcal{R}_0^\varepsilon(\pi - p^\varepsilon(t), \beta - \alpha^\varepsilon(t)), \end{aligned}$$

for every $(v, \eta, \pi, \beta) \in \mathcal{A}_0^{\mathrm{hard},\varepsilon}(w(t))$.

(2) Energy balance:

$$\begin{aligned} &\mathcal{Q}_0^{\mathrm{el},\varepsilon}(e^\varepsilon(t)) + \delta(\varepsilon)\, \mathcal{Q}_0^{\mathrm{hard},\varepsilon}(p^\varepsilon(t), \alpha^\varepsilon(t)) + \mathcal{V}_{\mathcal{R}_0^\varepsilon}(p, \alpha; 0, t) \\ &= \mathcal{Q}_0^{\mathrm{el},\varepsilon}(e^\varepsilon(0)) + \delta(\varepsilon)\, \mathcal{Q}_0^{\mathrm{hard},\varepsilon}(p^\varepsilon(0), \alpha^\varepsilon(0)) \\ &+ \int_0^t \int_\Omega \mathbb{C}^{\mathrm{red}}\left(\tfrac{x'}{\varepsilon}\right) e^\varepsilon(s) : \mathbf{E}\dot{w}(s)\, dx ds \end{aligned}$$

Two-scale convergence of measures

Definition

Let $\{\mu^h\}_{h>0}$ be a family in $\mathcal{M}_b(\omega)$ and consider $\mu \in \mathcal{M}_b(\omega \times \mathcal{Y})$. We say that $\{\mu^h\}$ two-scale weakly* converges to μ if

$$\int_\omega \chi\left(x', \frac{x'}{\varepsilon}\right) d\mu^h(x') \xrightarrow[\varepsilon\to 0]{} \int_{\omega\times\mathcal{Y}} \chi(x', y)\, d\mu(x', y),$$

for every $\chi \in C_0(\omega \times \mathcal{Y})$. We write

$$\mu^h(x') \xrightharpoonup{2-*} \mu(x', y) \quad \text{in } \mathcal{M}_b(\omega \times \mathcal{Y}).$$

Two-scale limits for symmetrized gradients and Hessians

Proposition

Let $\{v^\varepsilon\}_{\varepsilon>0}$ be a bounded family in $BD(\omega)$ such that $v^\varepsilon \overset{*}{\rightharpoonup} v$ weakly* in $BD(\omega)$. Then

$$\mathbf{E}v^\varepsilon \xrightarrow{2-*} \mathbf{E}_{x'}v \otimes \mathcal{L}^2_y + \mathbf{E}_y\mu \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$$

with

$$\begin{cases} \mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{R}^2) : E_y\mu \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}), \\ \mu(F \times \mathcal{Y}) = 0 \;\; \text{for every Borel set } F \subseteq \omega. \end{cases}$$

Proposition

Let $\{v^\varepsilon\}_{\varepsilon>0}$ be a bounded family in $BH(\omega)$ such that $v^\varepsilon \overset{*}{\rightharpoonup} v$ weakly* in $BH(\omega)$. Then

$$\nabla^2 v^\varepsilon \xrightarrow{2-*} \nabla^2_{x'}v \otimes \mathcal{L}^2_y + \nabla^2_y\kappa \quad \text{two-scale weakly* in } \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$$

with

$$\begin{cases} \kappa \in \mathcal{M}_b(\omega \times \mathcal{Y}) : \nabla^2_y\kappa \in \mathcal{M}_b(\omega \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}), \\ \kappa(F \times \mathcal{Y}) = 0 \;\; \text{for every Borel set } F \subseteq \omega, \end{cases}$$

Admissible configurations and energies ($\varepsilon = 0$)

The admissible two-scale configurations $\mathcal{A}_0^{\mathrm{hom}}(w)$ are the class of all triples

$$(u, E, P) \in (BD(\widetilde{\Omega}) \cap KL(\widetilde{\Omega})) \times L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}) \times \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}})$$

such that

$$u = w, \qquad E = Ew, \qquad P = 0 \qquad \text{on } (\widetilde{\Omega} \setminus \overline{\Omega}) \times \mathcal{Y},$$

satisfying:

$$\mathbf{E}u \otimes \mathcal{L}^2_y + \mathbf{E}_y \mu - x_3 \nabla^2_y \kappa = E\, \mathcal{L}^3_x \otimes \mathcal{L}^2_y + P \quad \text{in } \widetilde{\Omega} \times \mathcal{Y}$$

For $(u, E, P) \in \mathcal{A}_0^{\mathrm{hom}}(w)$ we now define

$$\mathcal{Q}_0^{\mathrm{hom}}(E) := \frac{1}{2} \int_{\Omega \times \mathcal{Y}} \mathbb{C}^{\mathrm{red}}(y) E(x,y) : E(x,y)\, dx dy.$$

The reduced dissipation potential

The set $K^{\rm red}(y) \subset \mathbb{M}^{2\times 2}_{\rm sym}$ represents the set of admissible stresses in the reduced problem and can be characterised as follows:

$$M \in K^{\rm red}(y) \quad \Longleftrightarrow \quad \begin{pmatrix} M_{11} & M_{12} & 0 \\ M_{12} & M_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}(\operatorname{tr} M) I_{3\times 3} \in K(y)$$

The reduced perfectly-plastic dissipation potential $\mathbf{R}^{\rm red} : \cup_i \mathcal{Y}_i \times \mathbb{M}^{2\times 2}_{\rm sym} \to [0,+\infty)$ is given by the support function of $K^{\rm red}(y)$, i.e.

$$\mathbf{R}^{\rm red}(y, M) := \sup_{\sigma \in K^{\rm red}(y)} \sigma : M \quad \text{for every } M \in \mathbb{M}^{2\times 2}_{\rm sym}.$$

For $i \neq j$, $\nu \in \mathbb{S}^1$, $\bar{c} \in \mathbb{R}^2$, $\hat{c} \in \mathbb{R}$ define

$$\mathbf{R}^{\rm red}_{ij}(\nu, \bar{c}, \hat{c}) := \inf_{C(\bar{c},\hat{c})} \left\{ \int_I \Big(\mathbf{R}^{\rm red}_i\big(\bar{c}^i \odot \nu + x_3 \hat{c}^i \, \nu \otimes \nu\big) + \mathbf{R}^{\rm red}_j\big(\bar{c}^j \odot \nu + x_3 \hat{c}^j \, \nu \otimes \nu\big) \Big) dx_3 \right\},$$

where the infimum is taken over the set

$$C(\bar{c}, \hat{c}) := \Big\{ (\bar{c}^i, \bar{c}^j, \hat{c}^i, \hat{c}^j) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} : \bar{c}^i + \bar{c}^j = \bar{c},\ \hat{c}^i + \hat{c}^j = \hat{c} \Big\}.$$

Non-local definition of dissipation energy

For any measure $P \in \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\rm sym})$ which has the form on the interface

$$P\lfloor\widetilde{\Omega} \times (\Gamma_{ij} \setminus S) = \left[\bar{c}(x', y) \odot \nu(y) + x_3 \hat{c}(x', y)\, \nu(y) \otimes \nu(y)\right] \zeta \otimes \mathcal{L}^1_{x_3},$$

we define $\mathbf{R}^{\rm red}(P) \in \mathcal{M}_b^+(\widetilde{\Omega} \times \mathcal{Y})$ by

$$\mathbf{R}^{\rm red}(P) := \sum_i \mathbf{R}_i^{\rm red}(P) + \sum_{i<j} \mathbf{R}_{ij}^{\rm red}(P),$$

where

$$\mathbf{R}_i^{\rm red}(P) := \mathbf{R}_i^{\rm red}\left(\frac{dP}{d|P|}\right)|P| \in \mathcal{M}_b^+(\widetilde{\Omega} \times \mathcal{Y}_i),$$
$$\mathbf{R}_{ij}^{\rm red}(P) := \mathbf{R}_{ij}^{\rm red}(\nu, \bar{c}, \hat{c})\, \zeta \otimes \mathcal{L}^1_{x_3} \in \mathcal{M}_b^+(\widetilde{\Omega} \times (\Gamma_{ij} \setminus S)).$$

Finally, we define the non-local definition of dissipation energy

$$\mathcal{R}_0^{\rm hom}(P) := \sum_i \int_{\widetilde{\Omega} \times \mathcal{Y}_i} \mathbf{R}_i^{\rm red}\left(\frac{dP}{d|P|}\right) d|P| + \sum_{i<j} \int_{\widetilde{\omega} \times (\Gamma_{ij} \setminus S)} \mathbf{R}_{ij}^{\rm red}(\nu(y), \bar{c}(x', y), \hat{c}(x', y))\, d\zeta.$$

The principle of maximum plastic work

Proposition

For every $\Sigma \in \mathcal{K}_0^{hom}$ and $(u, E, P) \in \mathcal{A}_0^{hom}(w)$ we have

$$\mathcal{R}_0^{hom}(P) \geq -\int_{\Omega\times\mathcal{Y}} \Sigma : E \, dx dy + \int_\omega \bar{\sigma} : E\bar{w} \, dx' - \frac{1}{12}\int_\omega \hat{\sigma} : \nabla^2 w_3 \, dx'.$$

A notion of two-scale quasistatic elasto-plastic evolution

The two-scale quasistatic evolution is a function

$$t \mapsto (u(t), E(t), P(t)) \in \mathcal{A}_0^{hom}(w(t))$$

which satisfies the following conditions for every $t \in [0, T]$:

(1) Global stability:

$$\mathcal{Q}^{hom}(E(t)) \leq \mathcal{Q}^{hom}(H) + \mathcal{R}_0^{\mathrm{hom}}(\Pi - P(t)),$$

for every $(v, H, \Pi) \in \mathcal{A}_\gamma^{hom}(w(t))$.

(2) Energy balance:

$$\mathcal{Q}^{hom}(E(t)) + \mathcal{V}_{\mathcal{R}_0^{\mathrm{hom}}}(P; 0, t)$$
$$= \mathcal{Q}^{hom}(E(0)) + \int_0^t \int_{\Omega \times \mathcal{Y}} \mathbb{C}(y) E(s) : E\dot{w}(s) \, dx dy ds.$$

Main result

Theorem

Let $t \mapsto w(t)$ be an absolutely continuous boundary datum of Kirchhoff-Love type and consider a compact sequence of initial data $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon, \alpha_0^\varepsilon) \in \mathcal{A}_0^{\mathrm{hard},\varepsilon}(w(0))$.

Let $t \mapsto (u^\varepsilon(t), e^\varepsilon(t), p^\varepsilon(t), \alpha^\varepsilon(t))$ be a ε-quasistatic evolution on $\widetilde{\Omega}$ with boundary datum $w(t)$ and the initial datum $(u_0^\varepsilon, e_0^\varepsilon, p_0^\varepsilon, \alpha_0^\varepsilon)$.

Then, there exists a two-scale quasistatic evolution $t \mapsto (u(t), E(t), P(t))$ for the boundary datum $w(t)$ such that (up to subsequences)

$$u^\varepsilon(t) \xrightharpoonup{*} u(t) \quad \text{weakly* in } BD(\widetilde{\Omega}),$$

$$e^\varepsilon(t) \xrightharpoonup{2} E(t) \quad \text{two-scale weakly in } L^2(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}),$$

$$(p^\varepsilon)^{1,2}(t) \xrightharpoonup{2-*} P(t) \quad \text{two-scale weakly* in } \mathcal{M}_b(\widetilde{\Omega} \times \mathcal{Y}; \mathbb{M}^{2\times 2}_{\mathrm{sym}}),$$

for every $t \in [0, T]$.

Thank you for your attention