Homogenisation of high-contrast integral operators

I. Velčić
University of Zagreb, Faculty of Electrical Engineering and Computing

joint work with A. Piantniski (UIT Narvik) and M. Cherdantsev (University of Cardiff)

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Spectrum of the limit (homogenised) operator

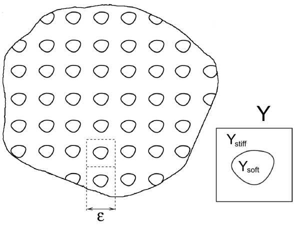
Differential operators with periodic coefficients

$$-\nabla \cdot a\,(x/\varepsilon)\,\nabla u_\varepsilon(x) = f(x)$$
$$a(y) \text{ is periodic}, \quad \varepsilon << 1$$

Homogenised problem:

$$\mathcal{A}_{\mathrm{hom}} u(x) = -\nabla \cdot a_{\mathrm{hom}} \nabla u(x) = f(x)$$

Spectrum of the limit operator $\mathrm{Sp}(\mathcal{A}_{\mathrm{hom}}) = \mathbb{R}^+$.

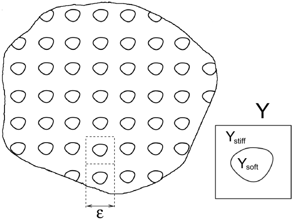


Operators with high-contrast coefficients

$$\mathcal{A}_\varepsilon := -\nabla \cdot a(x,\varepsilon)\nabla$$

$$a(x,\varepsilon) := \begin{cases} \varepsilon^2, & x \in S^\varepsilon_{\text{soft}} \\ a_1, & x \in S^\varepsilon_{\text{stiff}} \end{cases}$$

$$\mathcal{A}_\varepsilon u_\varepsilon + \lambda u_\varepsilon = f_\varepsilon$$



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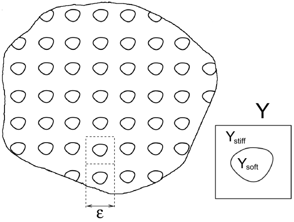

Homogenised operator is two-scale

$$\mathcal{A}_{\text{hom}} u + \lambda u = f, \quad \lambda > 0$$

$$u(x,y) = u_0(x) + v(x,y)$$

$$\begin{cases} -\nabla \cdot a_1^{\text{hom}} \nabla u_0 + \lambda(u_0 + \langle v \rangle) = \langle f \rangle, & x \in \mathbb{R}^n, \\ -\Delta_y v + \lambda(u_0 + v) = f, & y \in Y_0, \ \text{for fixed } x \in \mathbb{R}^n. \end{cases}$$

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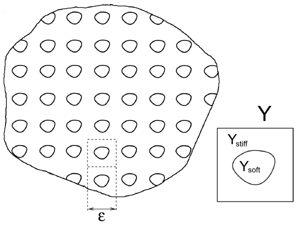

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Main order approximation to the solution of the original problem:

$$u(x, x/\varepsilon) = u_0(x) + v(x, x/\varepsilon)$$

Spectral problem for the homogenised operator

$$\mathcal{A}_{\mathrm{hom}}(u_0 + v) = \lambda(u_0 + v) \quad \Leftrightarrow \quad \begin{aligned} -\nabla \cdot a_1^{\mathrm{hom}} \nabla u_0(x) &= \lambda \left(u_0(x) + \langle v(x,\cdot) \rangle \right) \text{ in } \mathbb{R}^n, \\ -\Delta_y v(x,y) &= \lambda \left(u_0(x) + v(x,y) \right) \text{ in } Y_0. \end{aligned}$$

Band-gap spectrum in the high-contrast setting

Spectral problem for the homogenised operator

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$$v(x,y)=\lambda u_0(x)b_\lambda(y)$$

$$-\Delta_y b_\lambda-\lambda b_\lambda=1 \text{ in } Y_0, \quad b=0 \text{ on } \partial Y_0.$$

$$b_\lambda=(-\Delta_{Y_0}-\lambda)^{-1}\mathbf{1}_{Y_0}$$

Band-gap spectrum in the high-contrast setting

Spectral problem for the homogenised operator

$$\mathcal{A}_{\rm hom}(u_0+v)=\lambda(u_0+v) \quad \Leftrightarrow \quad \begin{array}{r} -\nabla\cdot a_1^{\rm hom}\nabla u_0(x)=\lambda\,(u_0(x)+\langle v(x,\cdot)\rangle) \ \text{ in } \mathbb{R}^n, \\ -\Delta_y v(x,y)=\lambda\,(u_0(x)\,+\,v(x,y)) \ \text{ in } Y_0. \end{array}$$

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$$-\nabla\cdot a_1^{\rm hom}\nabla u_0=\beta(\lambda)u_0$$

$$\beta(\lambda):=\lambda+\lambda^2\langle b_\lambda\rangle=\lambda\,+\,\lambda^2\sum_{j=1}^{\infty}\frac{\langle\varphi_j\rangle^2}{\nu_j-\lambda}$$

ν_j,φ_j - eigenvalues and eigenfunctions of $-\Delta_{Y_0}$.

Theorem

$$\mathrm{Sp}(\mathcal{A}_{\rm hom})=\{\lambda:\beta(\lambda)\geq 0\}\cup\mathrm{Sp}(-\Delta_{Y_0}).$$

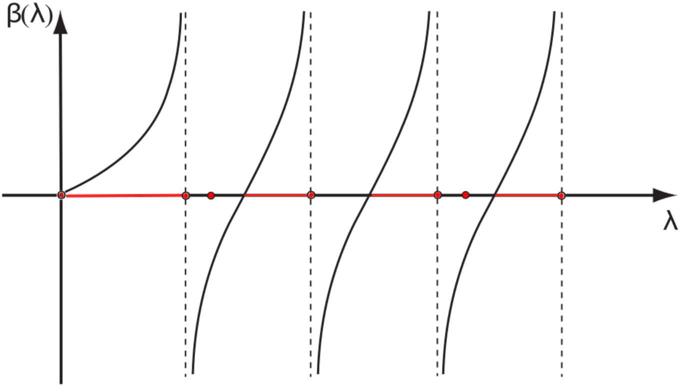

Figure: Spectrum of $\mathcal{A}_{\mathrm{hom}}$ is in red

Convergence of the spectra:

$\mathrm{Sp}(\mathcal{A}_\varepsilon) \to \mathrm{Sp}(\mathcal{A}_{\mathrm{hom}})$ in the sense of Hausdorff.
(only when the soft inclusions are disconnected!)

For a symmetric kernel $K(x,y)$ define operator $\mathcal{A} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$:

$$\mathcal{A}u(x) := 2\int_{\mathbb{R}^d} K(x,y)(u(x)-u(y))dy = f(x), \quad u \in L^2(\mathbb{R}^d).$$

$$\mathcal{A}u(x) = m(x)u(x) - 2\int_{\mathbb{R}^d} K(x,y)u(y)dy, \text{ where } m(x) := 2\int_{\mathbb{R}^d} K(x,y)dy.$$

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Weak formulation and bilinear form:

$$a(u,v) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)(u(y)-u(x))(v(y)-v(x))dydx = \int_{\mathbb{R}^d} f(x)v(x)dx, \quad \forall v \in L^2(\mathbb{R}^d).$$

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In bounded domain $S \subset \mathbb{R}^d$ with Dirichlet b.c.'s
Bilinear form is defined on $u, v \in L^2(S)$ (meaning $u, v = 0$ outside S)

$$\mathcal{A}u(x) = \mathbf{1}_S(x)\, 2\int_{\mathbb{R}^d} K(x,y)(u(x)-u(y))dy, \quad u \in L^2(S).$$

Diffusive scaling for convolution kernel.
Let $K(x,y) = a(y-x)$ for some symmetric $a(\xi)$, $\xi = y - x$, such that

$$a \geq 0, \quad a(\xi) \geq c_0 > 0, \text{ for } |\xi| < r,\, r > 0$$
$$a(\xi)(1+|\xi|^2) \in L^1(\mathbb{R}^d),$$

Define

$$\mathcal{A}_\varepsilon u(x) := \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a(\xi/\varepsilon)(u(x+\xi) - u(x))d\xi$$

What happens when $\varepsilon \to 0$?

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$$a_\varepsilon(u,v) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(\xi) \frac{(u(x+\varepsilon\xi) - u(x))}{\varepsilon} \frac{(v(x+\varepsilon\xi) - v(x))}{\varepsilon} d\xi dx$$

Assume that u, v are smooth: $u(x+\varepsilon\xi) - u(x) \approx \varepsilon\, \xi \cdot \nabla u(x)$.

$$\iint a(\xi)(\xi\cdot\nabla u(x))(\xi\cdot\nabla v(x))d\xi dx = \int \Big[\int a(\xi)\, \xi_i\xi_j\, d\xi\Big]\partial_i u(x)\partial_j v(x)dx = \int A\nabla u(x)\cdot\nabla v(x)$$

Diffusive scaling for convolution kernel.

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Mollification technique.

Theorem

Let $\{u_\varepsilon\}$ *be bounded in* $L^2(\Omega)$ *sequence, and*

$$C_1 a_\varepsilon(u,u)\le \int_{\mathbb{R}^d}\int_{B_r}\left|\frac{u_\varepsilon(x+\varepsilon\xi)-u_\varepsilon(x)}{\varepsilon}\right|^2 d\xi dx \le C_2 a_\varepsilon(u,u)\le M$$

Then $\exists$ *smooth, bounded in* $H^1(\mathbb{R}^d)$ *family* $\{w_\varepsilon\}$ *such that*

$$\|u_\varepsilon - w_\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\sqrt{M}\varepsilon, \quad \|w_\varepsilon\|_{H^1(\mathbb{R}^d)}\le C(\sqrt{M}+\|u_\varepsilon\|_{L^2(\mathbb{R}^d)}).$$

High-contrast periodic convolution type integral operators

Previous works in homogenisation of convolution type integral operators are by Braides, Chiado Piat, D'Elia, Piatnitski, Sloushch, Suslina, Zhizhina,...

$$\mathcal{A}_\varepsilon u := \frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Big[\Lambda_1\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon^2 \Lambda_0\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\Big](u(y)-u(x))dy$$

$$\Lambda_1(x,y) = w_0(x,y)1_{Y_1}(x)1_{Y_1}(y),$$

$$\Lambda_0(x,y) = w_1(x,y)(1-1_{Y_1}(x)1_{Y_1}(y)),$$

$$0 < \alpha_1 < w_i(x,y) < \alpha_2 < \infty \text{ are symmetric}$$

$$Y = Y_0 \cup Y_1 \text{ is the periodicity cell}$$

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Theorem (Extension Result)

For any $u \in L^2(\varepsilon Y_1^{\#})$, there exists an extension $\widetilde{u}_\varepsilon \in L^2(\mathbb{R}^d)$ (i.e. $\widetilde{u}_\varepsilon = u$ on $\varepsilon Y_1^{\#}$) such that

$$\int_{\{|x-y|\le\varepsilon r'\}} \left(\frac{\widetilde{u}_\varepsilon(y)-\widetilde{u}_\varepsilon(x)}{\varepsilon}\right)^2 dydx \le C \int_{Y_1^{\#}\times Y_1^{\#}\cap\{|x-y|\le\varepsilon r\}} \left(\frac{u(y)-u(x)}{\varepsilon}\right)^2 dydx,$$

$$\|\widetilde{u}_\varepsilon\|_{L^2(\mathbb{R}^d)} \le C\|u\|_{L^2(Y_1^{\#})}$$

Definition

We say that a bounded in $L^2(S)$ sequence u_ε converges weakly two-scale to $u \in L^2(S \times Y)$, $u_\varepsilon(x) \xrightharpoonup{2} u(x,y)$, if for all $\varphi \in C_0^\infty(S)$, $b \in C_{\text{per}}^\infty(Y)$

$$\lim_{\varepsilon \to 0} \int_S u_\varepsilon(x)\varphi(x)b\left(\frac{x}{\varepsilon}\right)\,dx = \int_S \int_Y u(x,y)\varphi(x)b(y)\,dydx.$$

u_ε converges strongly two-scale to $u \in L^2(S \times Y)$, $u_\varepsilon(x) \xrightarrow{2} u(x,y)$, if $u_\varepsilon(x) \xrightharpoonup{2} u(x,y)$ and

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^2(S)} = \|u\|_{L^2(S \times Y)}$$

The corrector problem:

$$\int_{\mathbb{R}^d} a(\xi)\Lambda_1(y, y+\xi)\left(\xi_i + \varphi^i(y+\xi) - \varphi^i(y)\right)\, d\xi = 0$$

Correctors $\varphi_i \in L^2(Y_1^{\#})$, $i = 1, \dots, d$ are unique up to an additive constant.

The homogenised matrix of the stiff component:

$$A_{ij}^{\mathrm{hom}} = \int_Y \int_{\mathbb{R}^d} a(\xi)\Lambda_1(y, y+\xi)\left(\xi_i + \varphi^i(y+\xi) - \varphi^i(y)\right)\xi_j\, d\xi\, dy,$$

Theorem

Let $f_\varepsilon \stackrel{2}{\rightharpoonup} (\stackrel{2}{\to}) f(x,y)$. *Then*

$$u_\varepsilon \stackrel{2}{\rightharpoonup} (\stackrel{2}{\to}) u_0 + v(x,y), \quad u_0 \in H_0^1(S),\ v \in L^2(\mathbf{R}^d \times Y),\ v = 0 \text{ on } Y_1.$$

where (u_0, v) *is the solution to the homogenised problem*

$$-\nabla \cdot A^{\mathrm{hom}} \nabla u_0(x) + \lambda\left(u_0(x) + \langle v(x,y)\rangle\right) = \langle f(x,y)\rangle$$

$$\mathbf{1}_{Y_0}(y) \int a(\xi)\Lambda_0(y, y+\xi)(v(x, y+\xi) - v(x,y))d\xi + \lambda(u_0(x) + v(x,y)) = f(x,y)$$

The homogenised operator

$$\mathcal{A}^{\text{hom}} = (\mathcal{A}_{\text{stiff}}, \mathcal{A}^{\#}_{\text{soft}})$$

$$\begin{aligned}
\mathcal{A}_{\text{stiff}}\, u_0 &= -\nabla \cdot A^{\text{hom}} \nabla u_0 \\
\mathcal{A}^{\#}_{\text{soft}}\, v(x,y) &= \mathbf{1}_{Y_0}(y) \int_{\mathbb{R}^d} a(\xi)\Lambda_0(y, y+\xi)(v(x,y+\xi) - v(x,y))d\xi \\
v(x,\cdot) &\in L^2_{\#}(Y_0) \quad \text{(i.e. } v = 0 \text{ on } Y_1 \text{ and is periodic in } y\text{)}
\end{aligned}$$

$$\mathcal{A}^{\text{hom}}(u_0, v) + \lambda(u_0, v) = f \quad \Leftrightarrow \quad \left\{ \begin{array}{c} \mathcal{A}_{\text{stiff}}\, u_0 + \lambda(u_0 + \langle v \rangle) = \langle f \rangle \\ \mathcal{A}^{\#}_{\text{soft}}\, v + \lambda(u_0 + v) = f \end{array} \right.$$

Spectrum of the homogenised operator

$$\mathcal{A}_{\text{stiff}}\, u_0 = \lambda(u_0 + \langle v\rangle) \tag{1}$$

$$\mathcal{A}^{\#}_{\text{soft}}\, v = \lambda(u_0 + v) \tag{2}$$

$$\mathrm{Sp}(\mathcal{A}^{\#}_{\text{soft}}) \subset \mathrm{Sp}(\mathcal{A}^{\text{hom}})$$

Spectrum of the homogenised operator

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β-function

Let $\mathcal{A}^{\#}_{\text{soft}}\, b_\lambda - \lambda b_\lambda = \mathbf{1}_{Y_0}$ i.e. $b_\lambda = (\mathcal{A}^{\#}_{\text{soft}} - \lambda)^{-1}\mathbf{1}_{Y_0}$

then $v = \lambda u_0 b_\lambda$ solves (2). Substituting this into (1) we have

$$\mathcal{A}_{\text{stiff}}\, u_0 = (\lambda + \lambda^2 \langle b_\lambda \rangle) u_0 =: \beta(\lambda) u_0$$

Theorem

$$\mathrm{Sp}(\mathcal{A}^{\text{hom}}) = \mathrm{Sp}(\mathcal{A}^{\#}_{\text{soft}}) \cup \{\lambda :\ \beta(\lambda) \in \mathrm{Sp}(\mathcal{A}_{\text{stiff}})\}$$

Spectral convergence

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In general not!

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Define

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Theorem

$$\lim_{\varepsilon \to 0} \mathrm{Sp}(\mathcal{A}_\varepsilon) = \mathrm{Sp}(\mathcal{A}_{\mathrm{soft}}) \cup \{\lambda : \ \beta(\lambda) \in \mathrm{Sp}(\mathcal{A}_{\mathrm{stiff}})\}$$

$$\mathrm{Sp}(\mathcal{A}_{\mathrm{soft},\mathbb{R}^d}) \subset \lim_{\varepsilon\to 0} \mathrm{Sp}(\mathcal{A}_{\mathrm{soft},\varepsilon^{-1}S})$$

$$\mathrm{Sp}(\mathcal{A}_{\mathrm{soft},\varepsilon^{-1}S}) \setminus \mathrm{Sp}(\mathcal{A}_{\mathrm{soft},\mathbb{R}^d}) = \text{“boundary layer spectrum”}$$

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Particular case: if $S = (0, l_1) \times \cdots \times (0, l_d)$, $l_i \in \mathbb{N}$, is a rectangular box and $\varepsilon = \frac{1}{N}, N \in \mathbb{N}$, then the limit spectrum $\lim_{\varepsilon \to 0} \mathrm{Sp}(\mathcal{A}_{\mathrm{soft},\varepsilon^{-1}S})$ is a union of the spectra of $\mathcal{A}_{\mathrm{soft},\mathbb{R}^d_{+,j}}$, where $\mathbb{R}^d_{+,j}$ are all d-dimensional hyper-octants of space.

Norm resolvent convergence for the whole space case via Cooper-Kamotski-Smyshlyaev approach

$$\frac{1}{\varepsilon^{d+2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\left(\frac{x-y}{\varepsilon}\right) \left[\Lambda_1 \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon^2 \Lambda_0 \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right] (u_\varepsilon(y) - u_\varepsilon(x))(\overline{v}(y) - \overline{v}(x)) dy dx$$

$$+ \int_{\mathbb{R}^d} u\overline{v} = \int_{\mathbb{R}^d} f\overline{v}$$

Apply scaled Gelfand transform $G_\varepsilon : L^2(\mathbb{R}^d) \to L^2(\Box^* \times \Box), \Box^* := [-\pi, \pi]^d$,

$$(G_\varepsilon f)(\theta, y) := \left(\frac{\varepsilon^2}{2\pi}\right)^{d/2} \sum_{n \in \mathbb{Z}^d} f(\varepsilon(y+n)) e^{-i\theta \cdot (y+n)}$$

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$$+\int\limits_{\mathbb{R}^d} u\overline{v} = \int\limits_{\mathbb{R}^d} f\overline{v}$$

Apply scaled Gelfand transform $G_\varepsilon : L^2(\mathbb{R}^d) \to L^2(\square^* \times \square), \square^* := [-\pi,\pi]^d$,

$$(G_\varepsilon f)(\theta, y) := \left(\frac{\varepsilon^2}{2\pi}\right)^{d/2} \sum_{n\in\mathbb{Z}^d} f(\varepsilon(y+n))e^{-i\theta\cdot(y+n)}$$

we obtain an equivalent family of problems for $u_\theta^\varepsilon(\cdot) := (G_\varepsilon u^\varepsilon)(\theta,\cdot)$ for $\theta \in \square^*$

$$\varepsilon^{-2}\int_\square\int_{\mathbb{R}^d} a(x-y)\Lambda_1(y,x)(e^{i\theta\cdot(x-y)}u_\theta^\varepsilon(x)-u_\theta^\varepsilon(y))\overline{(e^{i\theta\cdot(x-y)}v(x)-v(y))}dxdy$$
$$+\int_\square\int_{\mathbb{R}^d} a(x-y)\Lambda_0(y,x)(e^{i\theta(x-y)}u_\theta^\varepsilon(x)-u_\theta^\varepsilon(y))\overline{(e^{i\theta\cdot(x-y)}v(x)-v(y))}dxdy$$
$$+\int_\square u_\theta^\varepsilon(y)\overline{v(y)}dy = \int_\square f_\theta^\varepsilon(y)\overline{v(y)}dy, \quad \forall v \in L^2(\square),$$

We rewrite this expression in the form

$$\varepsilon^{-2} a_\theta(u_\theta^\varepsilon, v) + b_\theta(u_\theta^\varepsilon, v) = (f, v), \quad \forall v \in L^2(\square),$$

In the operator form we have $\quad (\mathcal{A}_\varepsilon^\theta + 1) u_\theta^\varepsilon = f$

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Theorem

Consider the problems

$$\varepsilon^{-2} a_\theta(u_\theta^\varepsilon, \widetilde{v}) + b_\theta(u_\theta^\varepsilon, \widetilde{v}) = (f, \widetilde{v}), \quad \forall \widetilde{v} \in L^2(\square),$$

$$\varepsilon^{-2} A^{\mathrm{hom}} \theta \cdot \theta \, z \widetilde{z} + b_\theta(z + v, \widetilde{z} + \widetilde{v}) = (f, \widetilde{z} + \widetilde{v}), \quad \forall \widetilde{z} + \widetilde{v} \in \mathbf{C} + L^2(Y_0)$$

Then

$$\| u_\theta^\varepsilon - (v + z) \|_{L^2(\square)} \le h(\varepsilon) \| f \|_{L^2(\square)}$$

In the operator form we have

$$\| (\mathcal{A}_\varepsilon^\theta + 1)^{-1} - (\mathcal{A}_\varepsilon^{\mathrm{hom},\theta} + 1)^{-1} \| \le h(\varepsilon)$$

Theorem

$$d_{H,[0,M]} \Big(\mathrm{Sp}(\mathcal{A}_\varepsilon^\theta), \mathrm{Sp}(\mathcal{A}_\varepsilon^{\mathrm{hom},\theta}) \Big) \le C(M) h(\varepsilon)$$

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Theorem

$$d_{H,[0,M]}\Big(\mathrm{Sp}(\mathcal{A}_\varepsilon^\theta), \mathrm{Sp}(\mathcal{A}_\varepsilon^{\mathrm{hom},\theta})\Big) \leq C(M)h(\varepsilon)$$

Here $h(\varepsilon) = o(\varepsilon^2)$ is related to the rate of decay of the second moment of the convolution kernel

$$g(r) := \int_{|\xi|>r} a(\xi)|\xi|^2 d\xi$$

Define

$$\mathcal{A}_{\text{soft}}^{\theta} v \quad = \quad \mathbf{1}_{Y_0} \int_{\mathbb{R}^d} a(\xi)\Lambda_0(y, y+\xi)(e^{i\theta\cdot\xi} v(y+\xi) - v(y))d\xi$$

$$(v \text{ is periodic, } e^{i\theta\cdot\xi} v(y+\xi) \text{ is } \theta\text{-quasiperiodic})$$

$$\varepsilon^{-2} a_\theta^h(z, \widetilde{z}) = A^{\text{hom}}\xi\cdot\xi\, z\widetilde{z} =: A_\xi^{\text{hom}}\, z\widetilde{z}, \text{ where } \xi := \frac{\theta}{\varepsilon}$$

Then

$$d_{H,[0,M]}\big(\operatorname{Sp}(\mathcal{A}_\varepsilon^{\text{hom},\theta}), \operatorname{Sp}(\mathcal{A}_{\text{soft}}^{\theta})\big) \leq C(M)\varepsilon^{2/3}, \quad |\theta| \geq \varepsilon^{2/3}.$$

$$d_{H,[0,M]}\Big(\operatorname{Sp}(\mathcal{A}_\varepsilon^{\text{hom},\theta}), \operatorname{Sp}(A_\xi^{\text{hom}} + \mathcal{A}_{\text{soft}}^{\#})\Big) \leq C(M)\varepsilon^{2/3}, \quad |\theta| \leq \varepsilon^{2/3}, \xi = \frac{\theta}{\varepsilon}.$$

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Then

$$d_{H,[0,M]}\big(\mathrm{Sp}(\mathcal{A}^{{\rm hom},\theta}_\varepsilon), \mathrm{Sp}(\mathcal{A}^\theta_{\rm soft})\big) \le C(M)\varepsilon^{2/3}, \quad |\theta|\ge\varepsilon^{2/3}.$$

$$d_{H,[0,M]}\Big(\mathrm{Sp}(\mathcal{A}^{{\rm hom},\theta}_\varepsilon),\, \mathrm{Sp}(\mathcal{A}^{\rm hom}_\xi+\mathcal{A}^{\#}_{\rm soft})\Big) \le C(M)\varepsilon^{2/3}, \quad |\theta|\le\varepsilon^{2/3},\, \xi=\frac{\theta}{\varepsilon}.$$

Observe that $\mathcal{A}_{\rm soft} = \int_\theta^\oplus \mathcal{A}^\theta_{\rm soft}$ and $\mathrm{Sp}(\mathcal{A}_{\rm soft}) = \cup_\theta\, \mathrm{Sp}(\mathcal{A}^\theta_{\rm soft})$

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Then

$$d_{H,[0,M]}\big(\mathrm{Sp}(\mathcal{A}_\varepsilon^{\text{hom},\theta}), \mathrm{Sp}(\mathcal{A}_{\text{soft}}^{\theta})\big) \le C(M)\varepsilon^{2/3}, \quad |\theta| \ge \varepsilon^{2/3}.$$

$$d_{H,[0,M]}\Big(\mathrm{Sp}(\mathcal{A}_\varepsilon^{\text{hom},\theta}),\, \mathrm{Sp}(\mathcal{A}_\xi^{\text{hom}} + \mathcal{A}_{\text{soft}}^{\#})\Big) \le C(M)\varepsilon^{2/3}, \quad |\theta| \le \varepsilon^{2/3},\, \xi = \frac{\theta}{\varepsilon}.$$

Observe that $\mathcal{A}_{\text{soft}} = \int_\theta^\oplus \mathcal{A}_{\text{soft}}^\theta$ and $\mathrm{Sp}(\mathcal{A}_{\text{soft}}) = \cup_\theta \mathrm{Sp}(\mathcal{A}_{\text{soft}}^\theta)$

Theorem

$$d_{H,[0,M]}\Big(\mathrm{Sp}(\mathcal{A}_\varepsilon),\, \{\lambda : \beta(\lambda) \in \mathrm{Sp}(\mathcal{A}_{\text{stiff}})\} \cup \mathrm{Sp}(\mathcal{A}_{\text{soft}})\Big) \le C(M)\max\{h(\varepsilon), \varepsilon^{2/3}\}$$

Recall that

$$\mathrm{Sp}(\mathcal{A}^{\text{hom}}) = \{\lambda : \beta(\lambda) \in \mathrm{Sp}(\mathcal{A}_{\text{stiff}})\} \cup \mathrm{Sp}(\mathcal{A}_{\text{soft}}^{\#}) \subset \{\lambda : \beta(\lambda) \in \mathrm{Sp}(\mathcal{A}_{\text{stiff}})\} \cup \mathrm{Sp}(\mathcal{A}_{\text{soft}})$$

THANK YOU!