Poroelastic plate model obtained by simultaneous homogenization and dimension reduction

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Homogenization of local and nonlocal operators and homogeneous geometric structures, UIT Narvik

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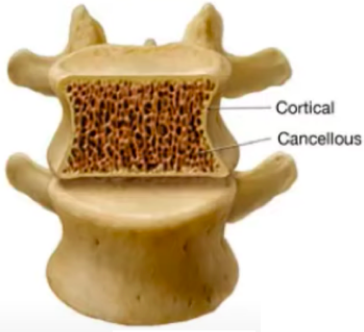
Outline

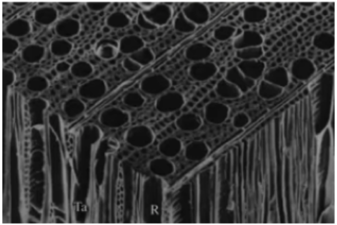
1. **Poroelastic plate model obtained by simultaneous homogenization and dimension reduction**
   - A (very) brief history of poroelasticity and motivation
   - Literature review
   - Main results
2. **References**

Poroelastic materials

are materials consisting of an elastic skeleton (the solid phase) and the pores saturated by a viscous fluid (the fluid phase). Some examples include:



Bone



Wood

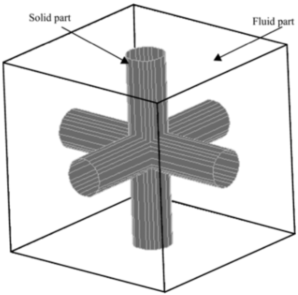
A (very) brief history of poroelasticity and motivation

- *Maurice Anthony Biot* (1941, Foundations of theory of poroelasticity). Biot, M. A. (1941). "General theory of three dimensional consolidation". Journal of Applied Physics. 12 (2): 155–164.
- *Andro Mikelić* was a pioneer in a mathematical analysis of poroelastic media.
  - **(i)** Mathematically rigurous derivation of Biot's systems, quasi-static Biot's equations of thin poroelastic plate;
  - **(ii)** Derivation of transmission laws at interface.

Interaction of fluid with a porous elastic structure

Let $\mathcal{Y} = (0,1)^3$ be the unit cell.

- Let $\mathcal{Y}_s$ (the solid part) be a closed subset of $\overline{\mathcal{Y}}$ and $\mathcal{Y}_f = \mathcal{Y} \setminus \mathcal{Y}_s$ (the fluid part).



- We make the following standard assumptions on $E_s = \bigcup_{k \in \mathbb{Z}^3} (k + \mathcal{Y}_s^k)$ and $E_f = \mathbb{R}^3 \setminus E_s$:
  - **(a)** $\mathcal{Y}_s$ is an open connected set of strictly positive measure, with a Lipschitz boundary;
  - **(b)** The interiors of E_s and E_f are open sets with Lipschitz boundary and E_s is connected.

Assume that $\Omega = (0, L)^3 \subset \mathbb{R}^3$ is covered with a regular mesh of size ε.

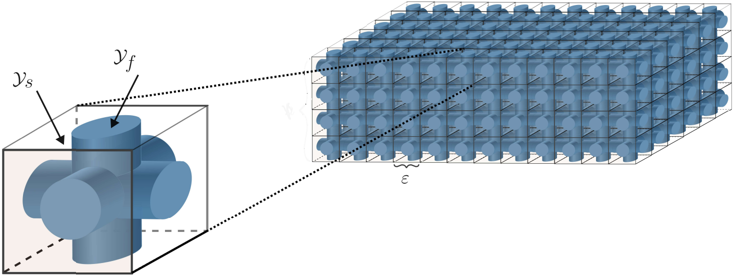


Figure: The fluid and solid phase

The fluid-solid interface is indicated by $\Gamma^\varepsilon := \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$. The domains Ω_s^ε and Ω_f^ε represent, respectively, the solid and fluid parts of a porous medium Ω.

The equations that describes this previous cases are given respectively

$$\left.\begin{aligned} &\varepsilon^m \rho_f \frac{\partial^2 \boldsymbol{u}^\varepsilon}{\partial t^2} - \mathrm{div}\sigma^{f,\varepsilon} = \rho_f \boldsymbol{F}, \\ &\nabla \cdot \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t} = 0, \end{aligned} \quad \text{in } \Omega_f^\varepsilon \times (0,T) \right\} \quad \text{(Stokes equation)}$$

$$\varepsilon^m \rho_s \frac{\partial^2 \boldsymbol{u}^\varepsilon}{\partial t^2} - \mathrm{div}\, \sigma^{s,\varepsilon} = \rho_s \boldsymbol{F} \quad \text{in } \Omega_s^\varepsilon \times (0,T), \quad \text{(elasticity equation)}$$

$$[\boldsymbol{u}^\varepsilon] = 0 \quad \text{on } \Gamma^\varepsilon \times (0,T) \quad \text{(displacement continuity at the interface)},$$

$$\sigma^{f,\varepsilon} = -p^\varepsilon I + 2\eta\varepsilon^r e\left(\frac{\partial \boldsymbol{u}^\varepsilon}{\partial t}\right) \quad \text{(fluid stress)},$$

$$\sigma^{s,\varepsilon} = Ae(\boldsymbol{u}^\varepsilon) \quad \text{(stress in solid)},$$

$$\sigma^{s,\varepsilon} \cdot \boldsymbol{n} = \sigma^{f,\varepsilon} \cdot \boldsymbol{n} \quad \text{on } \Gamma^\varepsilon \times (0,T), \quad \text{(continuity of normal stresses)}$$

$$\boldsymbol{u}^\varepsilon|_{\{t=0\}} = \partial_t \boldsymbol{u}^\varepsilon|_{\{t=0\}} = 0 \quad \text{on } \Omega, \quad \text{(initial condition)}$$

$$\{\boldsymbol{u}^\varepsilon, p^\varepsilon\} \quad \text{is periodic in } (x_1, x_2) \text{ with period } L.$$

- [GILBMIK2000] analysis of the above problem with
$$m = r = 0;$$
- [CLFRGILBMIK2001] analysis of the above problem with
$$m = 0 \quad \text{and} \quad r = 2;$$
- [MIKWHE2012] interface conditions between a poroelastic medium and an elastic body. Analysis of the above problem for $\Omega = (0, L)^3 \cup \Sigma \cup \Omega_{el}$, where $\Omega_{el} = (0, L)^2 \times (-L, 0)$, $\Sigma = (0, L)^2 \times \{0\}$ with
$$m = 1 \quad \text{and} \quad r = 2;$$
- [JAGMIK1996] interaction between porous media and fluid (2d)-derivation of contact conditions;

- [TAB1992] derivation of poroelastc plate model based on physical assumptions (plane stress, Kirchoff-Love ansatz, transverse fluid velocity is dominant);
- [MACMIK2015] derivation of poroelastic plate model starting from 3*d* Biot's equations for isotropic elastic tensor, using dimension reduction techniques;
- [GURWEB2022] analysis of poroelastic plate equation: existence and uniqueness of solution;
- [DuGunHouLee2002] linear fluid-structure interaction, existence, uniqueness, weak and strong solution;
- [GahJagNeu2022] regime $\varepsilon \sim h$ plate in fluid; the limit model is not of Biot's type;
- [Gah2024] in the regime $\varepsilon \sim h$ derivation of Biot's plate equations (different type of equation than in [MACMIK2015]);

Biot's bulk model

$\Omega^h = \{(x_1, x_2, x_3) \in \omega \times (-h/2, h/2)\}$, where the mid-surface ω is a bounded Lipschitz domain.

$$\sigma = 2Ge(\mathbf{u}) + (\frac{2\nu G}{1-2\nu} \operatorname{div} \mathbf{u} - \alpha p)I \text{ in } \Omega^h,$$

$$-G \triangle \mathbf{u} - \frac{G}{1-2\nu} \bigtriangledown \operatorname{div} \mathbf{u} + \alpha \bigtriangledown p = 0 \text{ in } \Omega^h, \quad (\operatorname{div} \sigma = 0),$$

$$\frac{\partial}{\partial t}(\gamma_G p + \alpha \operatorname{div} \mathbf{u}) - \frac{k}{\eta} \triangle p = 0 \text{ in } \Omega^h.$$

G-shear modulus, ν-Poisson ratio, α- effective stress coefficient, γ_G inverse of Biot's modulus, k-permeability, η-viscosity.

- the mean velocity of the fluid (velocity oscillates!) is proportional to the gradient of pressure (in case of absence of volume forces)-Darcy's law.
- the evolution model has memory effects!
- the constant k/η is additionally scaled with h^2.
- $\sigma \mathbf{n} = \mathcal{P}^{\pm h}$, $\partial_3 p = U^h$ at $x_3 = \pm h/2$.

Biot's plate model

$$G\Delta_{\hat{x}}\mathfrak{a} + \frac{G(1+\nu)}{1-\nu}\nabla_{\hat{x}}\mathrm{div}_{\hat{x}}\mathfrak{a} + \frac{\alpha(1-2\nu)}{1-\nu}\nabla_{\hat{x}}N + \sum_{j=1}^{2}(\mathcal{P}_j^{1/2} + \mathcal{P}_j^{-1/2})\mathbf{e}^j = 0,$$

$$\frac{G}{6(1-\nu)}\Delta_{\hat{x}}^2\mathfrak{b} + \alpha\frac{1-2\nu}{1-\nu}\Delta_{\hat{x}}\int_{-1/2}^{1/2} x_3 p^{eff}\, dx_3 =$$

$$\frac{1}{2}\sum_{i=1}^{2}\frac{\partial}{\partial x_i}(\mathcal{P}_i^{1/2} + \mathcal{P}_i^{-1/2}) + \mathcal{P}_3^{1/2} + \mathcal{P}_3^{-1/2}.$$

- $\mathfrak{b}(x_1, x_2, t)$ is the effective transverse displacement of the surface (scaled),
- $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{a}_j(x_1, x_2, t) - x_3\partial_j\mathfrak{b}$, $j = 1, 2$, are the effective in-plane solid displacements.

Biot's plate model

$$(\gamma_G + \frac{\alpha^2(1-2\nu)}{2G(1-\nu)})N = \frac{\alpha(1-2\nu)}{1-\nu}\,\mathrm{div}_{\hat{x}}\mathfrak{a},$$

$$(\gamma_G + \frac{\alpha^2(1-2\nu)}{2G(1-\nu)})\frac{\partial}{\partial t}(p^{eff}+N) - \frac{k}{\eta}\frac{\partial^2}{\partial x_3^2}(p^{eff}+N) = \alpha x_3 \frac{1-2\nu}{1-\nu}\frac{\partial}{\partial t}\Delta_{\hat{x}}\mathfrak{b},$$

$$\partial_3 p^{eff} = U^1 \text{ at } x_3 = \pm\frac{1}{2}.$$

- $p^{eff}(x_1, x_2, x_3, t)$ is the effective fluid pressure,
- $N = -\int_{-1/2}^{1/2} p^{eff}\, dx_3$ is the effective stress resultant due to the variation in pore pressure across the plate thickness.

Goal

Our goal is to justify limit model under simultaneous homogenization and dimension reduction (also the evolution). This enables us to

- understand the limit Darcy's law (which is not seen in continuum model);
- derive the contact of poroelastic and (poro)elastic plate and their relation to the microscopic model;
- understand the boundary conditions at the upper and lower boundary and their relation to the microscopic model.

In addition

- we will perform the computations for general elasticity tensor (isotropicity simplifies the limit model which then decouples, which is not true in general);
- we will obtain the limit model under weakest possible regularity of loads (in time), thus dealing with weak solutions analyzed in [DuGunHouLee2002].

Starting rescaled problem

Find $\boldsymbol{u}^h \in H^1(0,T;V^1)$, $p^h \in H^{-1}(0,T;L^2(\Omega))$ with $\frac{d^2\boldsymbol{u}^h}{dt^2} \in L^2(0,T;L^2(\Omega)^3)$ such that

$$\frac{d^2}{dt^2}\int_\Omega \eta^h \kappa^h \boldsymbol{u}^h(t)\varphi\,dx + \frac{d}{dt}\frac{\varepsilon^2}{h^4}\int_{\Omega_f^\varepsilon} 2e_h(\boldsymbol{u}^h(t)) : e_h(\varphi)\,dx$$
$$+\frac{1}{h^2}\int_{\Omega_s^\varepsilon} \mathbb{A}\left(\frac{\hat{x}}{\varepsilon},\frac{x_3}{\frac{\varepsilon}{h}}\right) e_h(\boldsymbol{u}^h(t)) : e_h(\varphi)\,dx - \frac{1}{h^2}\int_{\Omega_f^\varepsilon} p^h \operatorname{div}_h \varphi\,dx$$
$$= \int_\Omega \boldsymbol{F}^h\varphi\,dx, \quad \forall\varphi \in V^1, \quad \text{a.e. in } (0,T),$$

where

$$\kappa^h = \kappa_f^0 \chi_{\Omega_f^h} + \kappa_s^0 \chi_{\Omega_s^h}, \quad e_h := \operatorname{sym}\nabla_h, \quad \operatorname{div}_h := \operatorname{tr}\nabla_h, \quad \nabla_h := (\partial_1, \partial_2, \frac{1}{h}\partial_3).$$

Initial conditions

$$\boldsymbol{u}^h\big|_{\{t=0\}} = \frac{\partial \boldsymbol{u}^h}{\partial t}\bigg|_{\{t=0\}} = 0 \quad \text{on } \Omega = (0,L)^2 \times (-\frac{1}{2},\frac{1}{2}).$$

$$V^1 := H^1(\Omega;\mathbb{R}^3).\ \text{periodic in } x_1, x_2 \text{ direction}.$$

We consider the cases $\varepsilon = \varepsilon(h) \ll h$, $\eta^h \to 0$ and $\eta^h = 1$.

Assumptions

The tensor $\mathbb{A}$ is assumed to be uniformly positive definite on symmetric matrices:

$$\nu|\xi|^2 \leq \mathbb{A}(y)\xi : \xi \leq \nu^{-1}|\xi|^2 \quad \forall \xi \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \quad \forall y \in \mathcal{Y}_s.$$

Additional symmetries:

$$\mathbb{A}_{ijkl} = \mathbb{A}_{jikl} = \mathbb{A}_{klij}, \quad i, j, k, l \in \{1, 2, 3\}.$$

In the case $\eta^h \to 0$ we assume that $\boldsymbol{F}^h \in H^1(0, T; L^2(\Omega; \mathbb{R}^3))$

$$\left\| \pi_h \boldsymbol{F}^h \right\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C,$$

$$\pi_h \boldsymbol{F}^h := (h\boldsymbol{F}^h_1, h\boldsymbol{F}^h_2, \boldsymbol{F}^h_3) \xrightharpoonup{2} \boldsymbol{F}.$$

In the case when $\eta^h = 1$ we need only L^2 forces in the third component. We assume periodic boundary conditions in longitudinal direction and Neumann at the upper and lower boundary (fluid boundary conditions at transversal boundary?)

Assumptions

In general for deriving quasistatic model...

$$\varepsilon \partial_{tt} u - \Delta u = f, \quad u(0) = 0, \partial_t u(0) = 0; \quad u|_{\partial\Omega} = 0.$$

If we test it with $\partial_t u$ we obtain

$$\varepsilon \int_\Omega \partial_{tt} u \cdot \partial_t u - \int_\Omega \Delta u \cdot \partial_t u = \int_\Omega f \cdot \partial_t u.$$

We integrate over $[0, T]$ and obtain:

$$\varepsilon \|\partial_t u(T)\|_{L^2} + \|\nabla u(T)\|_{L^2} = \int_0^T \int_\Omega f \cdot \partial_t u.$$

In order to get the bounds either $\varepsilon \sim 1$ (use Young's and Grönwall's inequality) or

$$\int_0^T \int_\Omega f \cdot \partial_t u = - \int_0^T \int_\Omega \partial_t f \cdot u,$$

and then combine it with Young's and Poincare's inequality and use boundedness of f in H^1 and Grönwall's inequality.

The main result $\eta^h \to 0$

Theorem

The homogenized equations are given by:
Find $(\mathfrak{a}, \mathfrak{b}, p) \in L^2(0,T; H^1_\#(\omega)^2) \times L^2(0,T; H^2_\#(\omega)) \times L^2(0,T; L^2(\omega; H^1(I)))$ *such that*

$$\int_\omega \mathbb{A}^{\mathrm{hom}}(e_{\hat{x}}(\mathfrak{a}), \nabla^2_{\hat{x}}\mathfrak{b}) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla^2_{\hat{x}}\theta_3) d\hat{x} - \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{p} I : [\iota(e_{\hat{x}}(\boldsymbol{\theta}_*))]\, d\hat{x}$$
$$+ \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 p} I : \left[\iota(\nabla^2_{\hat{x}}\theta_3)\right]\, d\hat{x} = \int_\omega \overline{\langle \boldsymbol{F} \rangle_{\mathcal{Y}}} \cdot (\boldsymbol{\theta}_*, \theta_3)\, d\hat{x}$$
$$- \int_\omega \overline{\langle x_3 \boldsymbol{F}_* \rangle_{\mathcal{Y}}} \cdot \nabla_{\hat{x}}\theta_3\, d\hat{x} \quad \forall (\boldsymbol{\theta}_*, \theta_3) \in H^1_\#(\omega)^2 \times H^2_\#(\omega),$$
$$\frac{\partial}{\partial t} \int_\omega M_0 \overline{p\,\xi} d\hat{x} + \int_\omega K\, \overline{\partial_3 p\, \partial_3 \xi} d\hat{x} + \int_\omega (|\mathcal{Y}_f| I - \mathbb{B}^H) \overline{\xi} I : \iota\left(e_{\hat{x}}\left(\frac{\partial \mathfrak{a}}{\partial t}\right)\right) d\hat{x}$$
$$- \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 \xi} I : \iota\left(\nabla^2_{\hat{x}} \frac{\partial \mathfrak{b}}{\partial t}\right) d\hat{x} = 0, \quad \forall \xi \in V_1.$$

System has a unique solution.

The main result $\eta^h \to 0$

- The tensor $\mathbb{A}^{\mathrm{hom}}$ corresponds to the perforated domain;
- the constants M_0 and K are positive, $\mathbb{B}^H$ is symmetric;
- the system doesn't, in general, decouple on membrane and bending equations;
- we can still decouple the last equation by taking ξ independent of x_3 and the ones perpendicular to that ones;
- the limit effective fluid velocity is driven only by $\partial_3 p$, ie.,

$$\boldsymbol{v} = -(K_1, K_2, K)\partial_3 p.$$

Theorem

The homogenized equations are given by:
Find $(\mathfrak{a}, \mathfrak{b}, p) \in L^2(0,T; H^1_\#(\omega)^2) \times \left(L^2(0,T; H^2_\#(\omega)) \cap H^1(0,T; L^2(\omega))\right) \times L^2(0,T; L^2(\omega; H^1(I)))$ *such that*

$$\int_\omega \frac{\partial^2 \mathfrak{b}}{\partial t^2}\theta_3 + \int_\omega \mathbb{A}^{\mathrm{hom}}(e_{\hat{x}}(\mathfrak{a}), \nabla^2_{\hat{x}}\mathfrak{b}) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla^2_{\hat{x}}\theta_3) d\hat{x} - \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{p} I : [\iota(e_{\hat{x}}(\boldsymbol{\theta}_*))]\, d\hat{x}$$
$$+ \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 p} I : \left[\iota(\nabla^2_{\hat{x}}\theta_3)\right] d\hat{x} = \int_\omega \overline{\langle \boldsymbol{F} \rangle_{\mathcal{Y}}} \cdot (\boldsymbol{\theta}_*, \theta_3)\, d\hat{x}$$
$$- \int_\omega \overline{\langle x_3 \boldsymbol{F}_* \rangle_{\mathcal{Y}}} \cdot \nabla_{\hat{x}}\theta_3\, d\hat{x} \quad \forall (\boldsymbol{\theta}_*, \theta_3) \in H^1_\#(\omega)^2 \times H^2_\#(\omega),$$
$$\frac{\partial}{\partial t}\int_\omega M_0 \overline{p\,\xi} d\hat{x} + \int_\omega K\, \overline{\partial_3 p\, \partial_3 \xi} d\hat{x} + \int_\omega (|\mathcal{Y}_f| I - \mathbb{B}^H)\overline{\xi} I : \iota\left(e_{\hat{x}}\left(\frac{\partial \mathfrak{a}}{\partial t}\right)\right) d\hat{x}$$
$$- \int_\omega \left(|\mathcal{Y}_f| I - \mathbb{B}^H\right) \overline{x_3 \xi} I : \iota\left(\nabla^2_{\hat{x}} \frac{\partial \mathfrak{b}}{\partial t}\right) d\hat{x} = 0, \quad \forall \xi \in V_1.$$

System has a unique solution.

- there are no memory effects (bending plate model is long time evolution);
- Again system is a coupling of quasi-static and evolutionary equations (decoupling appears in the case of isotropicity). Eliminating $\mathfrak{a}$ in general causes spatial non-locality;
- System is hyperbolic-parabolic coupling.

We can treat different contacts of poroelastic and (poro)elastic plate. We assume finite number of different poroelastic cells. The most critical contacts are contacts across the thickness.

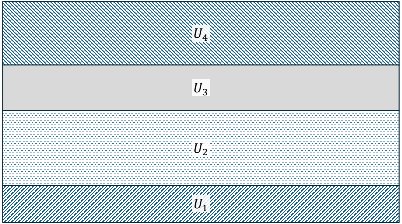
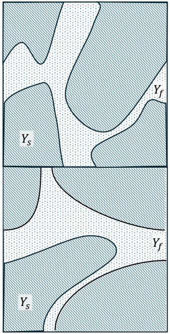
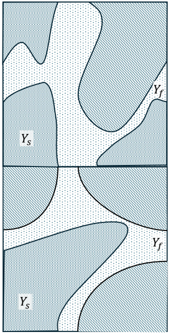


Figure: Figure illustrates the layer-like arrangement of the poroelastic plate when $\varepsilon(h) > 0$

The case of contacts-interface



(a)



(b)

Figure: (a) is an example of interface where there is no flow, while (b) is an example of interface such that there is a flow.

Remarks

- in the case of $\eta^h = 1$ we work with weak solution for fluid-elastic structure interaction analyzed in [DuGunHouLee2002];
- we can only guarantee that pressure is bounded in H^{-1} in time and L^2 in space variable (with bounds independent of h);
- the fundamental tool used for doing dimension reduction in elasticity is (modified) Griso's decomposition which completely characterizes the sequences of deformations with bounded symmetrized scaled gradients (see [Griso2005],[Vel14],[BuzCheVelZub22]);

Modified Griso's decomposition:

$$\limsup_{n\to\infty} \|e_h(\boldsymbol{u}^h)\|_{L^2} < \infty \iff$$

$$\boldsymbol{u}^h(x) = \begin{pmatrix} \mathfrak{a}(\hat{x}) \\ \frac{\mathfrak{b}}{h} \end{pmatrix} - x_3 \begin{pmatrix} \nabla_{\hat{x}}\mathfrak{b}(\hat{x}) \\ 0 \end{pmatrix} + \boldsymbol{A}^h \begin{pmatrix} x_1 \\ x_2 \\ hx_3 \end{pmatrix} + \boldsymbol{c}^h + \boldsymbol{\psi}^h(x),$$

$$\boldsymbol{A}^h \in \mathbb{R}^{3\times 3}_{\text{skew}}, \quad \boldsymbol{c}^h \in \mathbb{R}^3, \quad (\psi_1^h, \psi_2^h, h\psi_3^h) \to 0.$$

$$e_h(\nabla_h \boldsymbol{\psi}^h) = -x_3 \imath(\nabla^2_{\hat{x}}\varphi^h) + e_h(\nabla_h \tilde{\boldsymbol{\psi}}^h) + o^h.$$

$$\varphi^h \to 0 \ \text{ strongly in } H^1(\omega), \quad \tilde{\boldsymbol{\psi}}^h \to 0 \ \text{ strongly in } L^2(\Omega, \mathbb{R}^3),$$

$$\limsup_{n\to\infty} \left(\|\varphi^h\|_{H^2} + \|\nabla_h \tilde{\boldsymbol{\psi}}^h\|_{L^2} \right) < \infty, \ \|o^h\|_{L^2} \to 0.$$

Remarks

- additional regularity of the limit pressure $C(0,T;L^2(\Omega)) \cap L^2(0,T;L^2(\omega;H^1(I)))$ is obtained by considering the equations satisfied by the limit pressure;
- we also obtain that $(\mathfrak{a},\mathfrak{b}) \in C(0,T;H^1_\#(\omega)^2 \times H^2_\#(\omega))$ by analyzing the limit equations.
- for the limit equations, it is possible to define the mild solution (with semigroup approach) which doesn't have enough regularity to be the unique solution of the equations (thus we use Galerkin approximation);

Remarks

- semigroup approach can be used to approximate the solution with more regular solution (strong solution) and then to prove energy equality, by using stability estimate. Energy equality is used to prove strong converegence of solutions (under the assumption of strong convergence of the loads);
- the initial condition of pressure is encoded in the limit equations and is non-zero for the loads that satisfy $\boldsymbol{F}(0) \neq 0$;
- for the case of contacts we always obtain that the pressure is in $L^2(0, T; L^2(\omega; H^1(I_i)))$, for every (poro)elastic layer I_i, $i = 1, \ldots, m$. In order to obtain the model we need to characterize the jump of the trace of the pressure on every $\omega \times \partial I_i$;
- when there is a thin elastic layer between I_i and I_{i+1} (possibly vanishing in the limit) the interface condition is $\partial_3 p = 0$;
- when the condition of the Figure b) is satisfied the pressure is continuous in the x_3 direction;
- similarly, at the upper and lower boundary if we have thin elastic layer (possibly vanishing in the limit) the boundary condition is $\partial_3 p = 0$.

Remarks

- it is possible to add surface loads at the elastic part of the upper and down boundary causing the additional terms appearing at the right hand side of the limit equations;
- it is difficult to justify, from the microscopic point of view, any other condition for the pressure besides $\partial_3 p = 0$ at the upper and down boundary;
- it is impossible to impose Dirichlet boundary condition for the fluid (also for the solid!) at the upper and down boundary (see [CoMuPi94] for the discussion of different boundary conditions for the Stokes equation);
- strong convergence of solutions (with appropriate correctors) is proved only under the condition $\boldsymbol{F}(0) = 0$ (which is often assumed in the literature for the derivation of the limit equations also). This is reasonable, since it implies $\mathfrak{a}(0) = 0$, $\mathfrak{b}(0) = 0$ and $p(0) = 0$.
- no quantitative approach by now.

Existence result for quasi-static case

V, H separable Hilbert spaces such that

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

Find $u \in L^2(0, T; V)$ such that

$$\begin{cases} \frac{d}{dt}[\mathcal{B}_0 u] + \mathcal{A}_0 u = S \in L^2(0, T; V') \\ [\mathcal{B}_0 u](0) = \mathcal{B}_0 u_0 \in H. \end{cases}$$

Definition

The function $u \in L^2(0, T; V)$ such that

$$-\int_0^T \langle \mathcal{B}_0 u(t), v'(t)\rangle_H \, dt + \int_0^T {}_{V'}\langle \mathcal{A}_0 u(t), v(t)\rangle_V \, dt = \int_0^T {}_{V'}\langle S(t), v(t)\rangle_V \, dt + \langle \mathcal{B}_0 u_0, v(0)\rangle_H,$$

holds for all $v \in \{w \in L^2(0, T; V) \cap H^1(0, T; H) : w(T) = 0\}$ is called a weak solution.

Existence result for quasi-static case

The following is given in [GURWEB2022].

Assumption

We assume that

1. $\mathcal{A}_0 \in \mathcal{L}(V, V')$ *is monotone on* V, *i.e.*

$$_{V'}\langle \mathcal{A}_0 v, v\rangle_V \geq 0, \quad \forall v \in V;$$

2. $\mathcal{B}_0 \in \mathcal{L}(H)$ *is self-adjoint positive semidefinite on* H*;*
3. *There exist constants* $\lambda, c > 0$ *such that*

$$_{V'}\langle \mathcal{A}_0 v, v\rangle_V + \lambda \langle \mathcal{B}_0 v, v\rangle_H \geq c\|v\|_V^2, \quad \forall v \in V.$$

Theorem

The unique solution of the above abstract problem satisfies

$$\|u\|^2_{L^2(0,T;V)} \leq C(c, \lambda)\left[\|S\|^2_{L^2(0,T;V')} + \langle \mathcal{B}_0 u_0, u_0\rangle_H\right].$$

Corollary

If $\mathcal{B}_0$ *is coercive*

$$\|u\|^2_{C(0,T;H)} \leq C(c, \lambda)\left[\|S\|^2_{L^2(0,T;V')} + \langle \mathcal{B}_0 u_0, u_0\rangle_H\right].$$

Existence result for quasi-static case

- the existence proof is done by using the Lax-Milgram;
- with the semigroup approach we obtain that $u \in C(0, T; H)$;
- we take $H = L^2(\Omega)$, $V = L^2(\omega; H^1(I))$;

$$\|v\|_V := \|v\|_{L^2} + \|\partial_3 v\|_{L^2}, \quad {}_{V'}\langle \mathcal{A}_0 v_1, v_2 \rangle_V = \int_\Omega K \partial_3 v_1 \cdot \partial_3 v_2 \, dx;$$

- we use the following additive decomposition from the first equation:

$$(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{b})^{\boldsymbol{F}} + (\mathfrak{a}, \mathfrak{b})^p;$$

Existence result for quasi-static case

$$\int_\omega \mathbb{A}^{\mathrm{hom}}(\hat{x})(e_{\hat{x}}(\mathfrak{a}^p), \nabla^2_{\hat{x}}\mathfrak{b}^p) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla^2_{\hat{x}}\theta_3)d\hat{x} =$$
$$\int_\omega \int_I \left(|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)\right) p\, dx_3 : [\iota(e_{\hat{x}}(\boldsymbol{\theta}_*))]\ d\hat{x}$$
$$- \int_\omega \int_I \left(|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)\right) x_3 p\, dx_3 : \left[\iota(\nabla^2_{\hat{x}}\theta_3)\right]\ d\hat{x}, \quad \forall\, (\boldsymbol{\theta}_*, \theta_3)\,;$$

$$\int_\omega \mathbb{A}^{\mathrm{hom}}(\hat{x})(e_{\hat{x}}(\mathfrak{a}^{\boldsymbol{F}}), \nabla^2_{\hat{x}}\mathfrak{b}^{\boldsymbol{F}}) : (e_{\hat{x}}(\boldsymbol{\theta}_*), \nabla^2_{\hat{x}}\theta_3)d\hat{x} = \langle \boldsymbol{F}, (\boldsymbol{\theta}_*, \theta_3)\rangle, \quad \forall\, (\boldsymbol{\theta}_*, \theta_3)\,.$$

- we plug it in the second equation and obtain the form of the abstract problem, after appropriate definition of the operators.

Existence result for quasi-static case

$$\langle \mathcal{B}_0 p_1, p_2 \rangle_H := \int_{\Omega_p} M_0(x) p_1(x) \cdot p_2(x)\, dx$$
$$+ \int_\omega \int_I (|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)) p_2\, dx_3 : \iota\, (e_{\widehat{x}}(\mathfrak{a}^{p_1}))\, d\widehat{x}$$
$$- \int_\omega \int_I \left(|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)\right) x_3 p_2\, dx_3 : \iota \left(\nabla^2_{\widehat{x}} \mathfrak{b}^{p_1}\right)\, d\widehat{x},$$

$${}_{V'}\langle S(t), v \rangle_V := -\int_\omega \int_I (|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)) v\, dx_3 : \iota \left(e_{\widehat{x}}\left(\mathfrak{a}^{\partial_t \boldsymbol{F}(t)}\right)\right) d\widehat{x}$$
$$+ \int_\omega \int_I \left(|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)\right) x_3 v\, dx_3 : \iota \left(\nabla^2_{\widehat{x}} \mathfrak{b}^{\partial_t \boldsymbol{F}(t)}\right)\, d\widehat{x},$$

$$\langle \mathcal{B}_0 p_0, h \rangle_H := \int_\omega \int_I (|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)) h\, dx_3 : \iota \left(e_{\widehat{x}}\left(\mathfrak{a}^{\boldsymbol{F}(0)}\right)\right) d\widehat{x}$$
$$- \int_\omega \int_I \left(|\mathcal{Y}_f(x)|\mathbb{I} - \mathbb{B}^H(x)\right) x_3 h\, dx_3 : \iota \left(\nabla^2_{\widehat{x}} \mathfrak{b}^{\boldsymbol{F}(0)}\right)\, d\widehat{x}.$$

Energy-type equality

The following energy equality can be proved:

$$\frac{1}{2}\int_\omega \mathbb{A}^{\mathrm{hom}}(\widehat{x})(e_{\widehat{x}}(\mathfrak{a}(t)),\nabla^2_{\widehat{x}}\mathfrak{b}(t))) : (e_{\widehat{x}}(\mathfrak{a}(t)),\nabla^2_{\widehat{x}}\mathfrak{b}(t))d\widehat{x} + \frac{1}{2}\int_\Omega M_0(x)p^2(t)dx$$
$$+\int_0^t\int_\Omega K|\partial_3 p|^2 dx\,dt = \langle \boldsymbol{F}(t),(\mathfrak{a}(t),\mathfrak{b}(t))\rangle - \langle \boldsymbol{F}(0),(\mathfrak{a}(0),\mathfrak{b}(0))\rangle$$
$$-\int_0^t \langle \partial_t \boldsymbol{F}(\tau),(\mathfrak{a}(\tau),\mathfrak{b}(\tau))\rangle\, d\tau + \frac{1}{2}\int_\Omega M_0(x)(p_0)^2\,dx,$$

- one would like to use $\boldsymbol{\theta}_* = \partial_t\mathfrak{a}$, $\theta_3 = \partial_t\mathfrak{b}$ in the first equation and $\xi = p$ in the second and sum them; however we don't have enough regularity;
- for regular enough initial condition and loads, the solution is also regular and thus we have energy-type equality;
- for the general case use an approximation of the initial conditions and loads and the stability estimate.

- we can interpret the solution as a mild solution via semigroup approach, but we again don't obtain enough regularity of the (unique) solution;
- we use Galerkin approximation to obtain the solution and the stability estimate;
- to conclude energy-type equality one uses approximation of the solution with regular one and semigroup approach;
- from the first equation we need to express $\mathfrak{a}$ via additive decomposition:

$$\mathfrak{a} = \mathfrak{a}^{\boldsymbol{F}} + \mathfrak{a}^{p} + \mathfrak{a}^{\mathfrak{b}};$$

and then consider the problem with unknowns $(\mathfrak{b}, \partial_t \mathfrak{b}, p)$.

References I

W. JÄGER AND A. MIKELIĆ On the boundary conditions at the contact interface between a porous medium and a free fluid *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, Vol. 23, No. 3 (1996), 403–465.

R.P. GILBERT, A. MIKELIĆ, Homogenizing the acoustic properties of the seabed, part I, *Nonlinear Anal.* TMA **40** (2000), 185-212.

TH. CLOPEAU, J. L. FERRÍN, R.P. GILBERT, A. MIKELIĆ, Homogenizing the acoustic properties of the seabed, part II, *Math. Comput. Modelling* **33** (2001), 821-841.

A. MIKELIĆ, M. F. WHEELER, On the interface law between a deformable porous medium containing a viscous fluid and an elastic body, *Math. Models Methods Appl. Sci.* Vol. 22, No. 11 (2012).

L.A. TABER,Theory for Transverse Deflection of Poroelastic Plates, *J. Appl. Mech.*, 59 (1992). 628–634

- A. Marciniak-Czochra and A. Mikelić A rigorous derivation of the equations for the clamped Biot-Kirchhoff-Love poroelastic plate, *Arch. Ration. Mech. Anal.*, Vol. 215, No. 3, (2015), 1035–1062.
- E. Gurvich and J.T. Webster Weak solutions for a poro-elastic plate system *Appl. Anal.* Vol. 101, No. 5 (2022), 1617–1636.
- Q. Du, M.D. Gunzburger, L.S. Hou and J. Lee Analysis of a linear fluid-structure interaction problem *Discrete Contin. Dyn. Syst. A*, Vol. 9, No. 3 (2003), 633–650.
- M. Gahn, W. Jäger and M. Neuss-Radu Derivation of Stokes-plate-equations modeling fluid flow interaction with thin porous elastic layers *Appl. Anal.*, Vol. 101, No. 12 (2022), 4319-4348.
- M. Gahn Derivation of a Biot-Plate-System for a thin poroelastic layer *Preprint*.

References III

C. Conca, F. Murat, O. Pironneau. The Stokes and Navier-Stokes equations with boundary conditions involving the pressure, *Japan. J. Math.* **20** (1994) (2),279–318.

G. Griso, Asymptotic behavior of structures made of plates, *Anal. Appl.* **3** (2005), 325–356.

I. Velčić. On the general homogenization of von Kármán plate equations from 3D nonlinear elasticity, *Anal. Appl. (Singap.)* **15** (1) (2017), 1–49.

M. Bužančić, K. Cherednichenko, I. Velčić, J. Žubrinić, Spectral and evolution analysis of composite elastic plates with high contrast, *J. Elasticity* **152** (2022), 79–177.

Thank you for your attention!