An Operator-Asymptotic Approach to Periodic Homogenization for Equations of Linearized Elasticity

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Outline of the talk

Periodic homogenization of elliptic PDE

Fix $\varepsilon > 0$ a small parameter.

Oscillatory periodic heterogeneous media

- Matrix-valued function $\mathbb{A}_\varepsilon(x)$ of material coefficients.
- $\mathbb{A}_\varepsilon(x) = \mathbb{A}(\frac{x}{\varepsilon})$.
- $\mathbb{A}$ is **Y-periodic in $\mathbb{R}^d$**, $Y = [0,1]^d$.

The matrix $\mathbb{A}$ is symmetric, and $\exists \alpha, \beta > 0$, such that:

$$\alpha|\xi|^2 \leqslant \mathbb{A}(x)\xi \cdot \xi \leqslant \beta|\xi|^2, \quad \forall x, \xi \in \mathbb{R}^d.$$

The associated elliptic operator:

$$\mathcal{A}_\varepsilon \boldsymbol{u} := -\operatorname{div}\left(\mathbb{A}_\varepsilon(x)\nabla \boldsymbol{u}\right), \quad \mathcal{D}\left(\mathcal{A}_\varepsilon\right) \subset H^1(\mathbb{R}^d).$$

What happens when $\varepsilon \to 0$? Approximation properties for the stationary case, parabolic and hyperbolic evolution, the spectrum?

Periodic homogenisation of elliptic PDE

Resolvent problem

Let $\boldsymbol{f}_\varepsilon \in L^2(\mathbb{R}^d)$. Find $\boldsymbol{u}_\varepsilon \in H^1(\mathbb{R}^d)$ such that:

$$\mathcal{A}_\varepsilon \boldsymbol{u}_\varepsilon + \boldsymbol{u}_\varepsilon = \boldsymbol{f}_\varepsilon, \quad \text{on } \mathbb{R}^d,$$

The matrix of homogenised material coefficients:

$$\mathbb{A}_0 \xi \cdot \xi := \min_{\varphi \in H^1_\#(Y)} \int_Y \mathbb{A}(y)\,(\xi + \nabla\varphi) \cdot (\xi + \nabla\varphi)\,,$$

$H^1_\#(Y) := \{\boldsymbol{u} \in H^1(Y), \boldsymbol{u} \text{ is } Y- \text{ periodic }\}$.

$$\mathcal{A}_0 \boldsymbol{u} := -\operatorname{div}\left(\mathbb{A}_0 \nabla \boldsymbol{u}\right), \quad \mathcal{D}(\mathcal{A}_0) = H^2(\mathbb{R}^d).$$

Resolvent problem for the homogenised operator

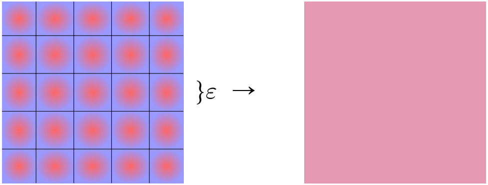
Let $\boldsymbol{f} \in L^2(\mathbb{R}^d)$. Find $\boldsymbol{u} \in H^1(\mathbb{R}^d)$ such that:

$$\mathcal{A}_0 \boldsymbol{u} + \boldsymbol{u} = \boldsymbol{f}, \quad \text{on } \mathbb{R}^d,$$

Qualitative and quantitative results

Qualitative results for $\boldsymbol{u}_\varepsilon \to \boldsymbol{u}$:

$$(\mathcal{A}_\varepsilon + I)^{-1} f_\varepsilon \to (\mathcal{A}_0 + I)^{-1} f, \quad \forall f_\varepsilon \to f.$$



Quantitative results are given with **the norm-resolvent estimates** (Birman, Suslina 2001., 2005., 2006., ...):

Quantitative result

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}_0 + I)^{-1}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}_0 + I)^{-1} - \varepsilon \mathcal{R}_{\text{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leqslant C\varepsilon,$$

$$\|(\mathcal{A}_\varepsilon + I)^{-1} - (\mathcal{A}_0 + I)^{-1} - \varepsilon \widehat{\mathcal{R}}_{\text{corr}}(\varepsilon)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leqslant C\varepsilon^2,$$

Periodic homogenisation of elliptic PDE

Methods: (capable of producing norm-resolvent estimates)

Periodic unfolding by Griso, Birman-Suslina spectral germ approach, Zhikov-Pastukhova shift method, Floquet-Bloch analysis by Zhikov, Conca, Vanninathan, Ganesh, refinement of the two-scale expansion method by Kenig, Lin, Shen, approaches by Waurick, Cooper, Kamotski, Smyshlyaev, Cherednichenko, D'Onofrio...

- Operator asymptotic approach used for simultaneous homogenisation and dimension reduction: Cherednichenko, Velčić (2021. thin heterogeneous plates), Cherednichenko, Velčić, Ž. (2023. thin heterogeneous rods)

The operator of linearized elasticity

Tensor of material coefficients $\mathbb{A}$:

- $\mathbb{A} : \mathbb{R}^3 \to \mathbb{R}^{3\times3\times3\times3}$ is $\mathbb{Z}^3-$periodic.
- $\mathbb{A}$ is uniformly (in y) positive definite on symmetric matrices: There exists $\nu > 0$ such that

$$\nu|\boldsymbol{\xi}|^2 \leqslant \mathbb{A}(y)\boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \frac{1}{\nu}|\boldsymbol{\xi}|^2, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^{3\times3}_{\mathsf{sym}}, \quad \forall y \in Y.$$

- The tensor $\mathbb{A}$ satisfies the following material symmetries:

$$\mathbb{A}^{ik}_{jl} = \mathbb{A}^{jk}_{il} = \mathbb{A}^{ki}_{lj}, \quad i,j,k,l \in \{1,2,3\}.$$

- The coefficients of $\mathbb{A}$ satisfy $\mathbb{A}^{ik}_{jl} \in L^\infty(Y)$, where $i,j,k,l \in \{1,2,3\}$.

We shall write $\mathbb{A}_\varepsilon = \mathbb{A}(\frac{\cdot}{\varepsilon})$.

Definition

The operator $\mathcal{A}_\varepsilon \equiv (\operatorname{sym}\nabla)^*\mathbb{A}_\varepsilon\,(\operatorname{sym}\nabla)$ is the operator on $L^2(\mathbb{R}^3;\mathbb{C}^3)$ defined through its corresponding sesquilinear form a_ε with form domain $H^1(\mathbb{R}^3;\mathbb{C}^3)$ and action

$$a_\varepsilon(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathbb{R}^3} \mathbb{A}\left(\frac{x}{\varepsilon}\right) \operatorname{sym}\nabla\boldsymbol{u}(x) : \overline{\operatorname{sym}\nabla\boldsymbol{v}(x)}\, dx, \quad \boldsymbol{u},\boldsymbol{v} \in \mathcal{D}(a_\varepsilon) = H^1(\mathbb{R}^3;\mathbb{C}^3).$$

The operator of linearized elasticity

Definition (Homogenized tensor)

Homogenized tensor $\mathbb{A}^{\mathrm{hom}} \in \mathbb{R}^{3\times3\times3\times3}$ is defined by

$$\mathbb{A}^{\mathrm{hom}}\boldsymbol{\xi} : \boldsymbol{\zeta} = \int_Y \mathbb{A}(y)\left(\boldsymbol{\xi} + \operatorname{sym}\nabla \boldsymbol{u}^{\boldsymbol{\xi}}\right) : \boldsymbol{\zeta}\, dy, \quad \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{R}^{3\times3}_{\mathsf{sym}},$$

where the *corrector term* $\boldsymbol{u}^{\boldsymbol{\xi}} \in H^1_\#(Y;\mathbb{R}^3)$ is the unique solution of the cell-problem:

$$\begin{cases} \int_Y \mathbb{A}\left(\boldsymbol{\xi} + \operatorname{sym}\nabla \boldsymbol{u}^{\boldsymbol{\xi}}\right) : \operatorname{sym}\nabla \boldsymbol{v}\, dy = 0, \quad \forall \boldsymbol{v} \in H^1_\#(Y;\mathbb{R}^3), \\ \int_Y \boldsymbol{u}^{\boldsymbol{\xi}} = 0. \end{cases}$$

Definition (Homogenized operator)

$$\mathcal{A}^{\mathrm{hom}} \equiv (\operatorname{sym}\nabla)^* \mathbb{A}^{\mathrm{hom}} (\operatorname{sym}\nabla),$$

with domain $\mathcal{D}\left(\mathcal{A}^{\mathrm{hom}}\right) = H^2(\mathbb{R}^3;\mathbb{C}^3)$. Its corresponding form is given by

$$a^{\mathrm{hom}}(\boldsymbol{u},\boldsymbol{v}) = \int_{\mathbb{R}^3} \mathbb{A}^{\mathrm{hom}} \operatorname{sym}\nabla \boldsymbol{u} : \overline{\operatorname{sym}\nabla \boldsymbol{v}}\, dy, \quad \boldsymbol{u},\boldsymbol{v} \in \mathcal{D}(a^{\mathrm{hom}}) := H^1(\mathbb{R}^3,\mathbb{C}^3).$$

Norm-resolvent estimates

Theorem

There exists $C > 0$ independent of the period of material oscillations $\varepsilon > 0$ such that the following norm-resolvent estimates hold:

- $L^2 \to L^2$ ***estimate.***

$$\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1} \right\|_{L^2(\mathbb{R}^3,\mathbb{R}^3) \to L^2(\mathbb{R}^3,\mathbb{R}^3)} \leqslant C\varepsilon.$$

- $L^2 \to H^1$ ***estimate.***

$$\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1} - \mathcal{R}^{\varepsilon}_{\mathrm{corr},1} \right\|_{L^2(\mathbb{R}^3,\mathbb{R}^3) \to H^1(\mathbb{R}^3,\mathbb{R}^3)} \leqslant C\varepsilon.$$

- ***Higher-order*** $L^2 \to L^2$ ***estimate.***

$$\left\| (\mathcal{A}_\varepsilon + I)^{-1} - \left(\mathcal{A}^{\mathrm{hom}} + I\right)^{-1} - \mathcal{R}^{\varepsilon}_{\mathrm{corr},1} - \mathcal{R}^{\varepsilon}_{\mathrm{corr},2} \right\|_{L^2(\mathbb{R}^3,\mathbb{R}^3) \to L^2(\mathbb{R}^3,\mathbb{R}^3)} \leqslant C\varepsilon^2.$$

Here, $\mathcal{R}^{\varepsilon}_{\mathrm{corr},1}$ and $\mathcal{R}^{\varepsilon}_{\mathrm{corr},2}$ are the corrector operators.

Strategy for the proof

- Use Gelfand transform in order to decompose the operator in the direct integral of operators with discrete spectrum defined on invariant infinitesimal subspaces
- Provide the estimates on the eigenvalues of these operators
- Identify the limit operator, provide norm-resolvent estimates by performing asymptotic expansion of the resolvent w.r.t. the quasimomentum, keeping in mind the scaling of the spectrum.
- Combine the optimal estimates into one uniform norm-resolvent estimate depending only on ε, by using functional calculus. Bring the fiber-wise estimate back to the full space.

Another important question: Can one obtain

$$(\mathcal{A}_\varepsilon + I)^{-1} = \left(\mathcal{A}^{\text{hom}} + I\right)^{-1} + \mathcal{R}^\varepsilon_{\text{corr},1} + \mathcal{R}^\varepsilon_{\text{corr},2} + \mathcal{R}^\varepsilon_{\text{corr},3} + \dots,$$

so that the error is of the order ε^n?

Gelfand transform

To study periodic functions and operators, we will make use of the Gelfand transform $\mathcal{G}$. This is defined as follows:

$$\mathcal{G} : L^2(\mathbb{R}^3; \mathbb{C}^3) \to L^2(Y'; L^2(Y; \mathbb{C}^3)) =: \int_{Y'}^{\oplus} L^2(Y; \mathbb{C}^3) d\chi$$

$$(\mathcal{G}\boldsymbol{u})(y, \chi) := \frac{1}{(2\pi)^{3/2}} \sum_{n \in \mathbb{Z}^3} e^{-i\chi\cdot(y+n)} \boldsymbol{u}(y+n), \quad y \in Y, \quad \chi \in Y',$$

The Gelfand transform $\mathcal{G}$ is a unitary operator

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{L^2(\mathbb{R}^3;\mathbb{C}^3)} = \langle \mathcal{G}\boldsymbol{u}, \mathcal{G}\boldsymbol{v} \rangle_{L^2(Y;L^2(Y';\mathbb{C}^3))}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in L^2(\mathbb{R}^3; \mathbb{C}^3),$$

and the inversion formula is given by

$$\boldsymbol{u}(x) = \frac{1}{(2\pi)^{3/2}} \int_{Y'} e^{i\chi\cdot x} (\mathcal{G}\boldsymbol{u})(x, \chi) d\chi, \quad x \in \mathbb{R}^3,$$

Scaled Gelfand transform

To deal with the setting of highly oscillating material coefficients, we will consider a scaled version of the Gelfand transform, denoted by $\mathcal{G}_\varepsilon$, where $\varepsilon > 0$. This is defined by:

$$\mathcal{G}_\varepsilon : L^2(\mathbb{R}^3; \mathbb{C}^3) \to L^2(Y'; L^2(Y; \mathbb{C}^3)) = \int_{Y'}^{\oplus} L^2(Y; \mathbb{C}^3) d\chi,$$

$$(\mathcal{G}_\varepsilon \boldsymbol{u})(y, \chi) := \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n \in \mathbb{Z}^3} e^{-i\chi\cdot(y+n)} \boldsymbol{u}(\varepsilon(y+n)), \quad y \in Y, \quad \chi \in Y'.$$

The inversion formula for $\mathcal{G}_\varepsilon$:

$$\boldsymbol{u}(x) = \frac{1}{(2\pi\varepsilon)^{3/2}} \int_{Y'} e^{i\chi\cdot\frac{x}{\varepsilon}} (\mathcal{G}_\varepsilon \boldsymbol{u}) \left(\frac{x}{\varepsilon}, \chi\right) d\chi, \quad x \in \mathbb{R}^3.$$

$$\mathcal{G}_\varepsilon (\operatorname{sym} \nabla \boldsymbol{u})(y, \chi) = \frac{1}{\varepsilon} \left(\operatorname{sym} \nabla_y (\mathcal{G}_\varepsilon \boldsymbol{u}) + i X_\chi (\mathcal{G}_\varepsilon \boldsymbol{u})\right),$$

where

$$X_\chi \boldsymbol{u} = \operatorname{sym}(\boldsymbol{u} \otimes \chi) = \operatorname{sym}\left(\boldsymbol{u}\chi^\top\right)$$

Operator fibres

Definition

The operator $\mathcal{A}_\chi \equiv (\operatorname{sym}\nabla + iX_\chi)^* \mathbb{A} (\operatorname{sym}\nabla + iX_\chi)$ on $L^2(Y;\mathbb{C}^3)$ is defined through sesquilinear form a_χ with form domain $H^1_\#(Y;\mathbb{C}^3)$ and action

$$a_\chi(\boldsymbol{u},\boldsymbol{v}) = \int_Y \mathbb{A}(y)(\operatorname{sym}\nabla + iX_\chi)\boldsymbol{u}(y) : \overline{(\operatorname{sym}\nabla + iX_\chi)\boldsymbol{v}(y)}\, dy, \quad \boldsymbol{u},\boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3).$$

Definition

For each $\chi \in Y'$, set $\mathcal{A}^{\text{hom}}_\chi \in \mathbb{C}^{3\times 3}$ to be the constant matrix satisfying

$$\left\langle \mathcal{A}^{\text{hom}}_\chi \boldsymbol{c}, \boldsymbol{d} \right\rangle_{\mathbb{C}^3} = \int_Y \mathbb{A}\left(\operatorname{sym}\nabla \boldsymbol{u_c} + iX_\chi \boldsymbol{c}\right) : \overline{iX_\chi \boldsymbol{d}}, \qquad \forall \boldsymbol{c},\boldsymbol{d} \in \mathbb{C}^3,$$

where $\boldsymbol{u_c} \in H^1_\#(Y;\mathbb{C}^3)$ is the unique solution of

$$\begin{cases} \int_Y \mathbb{A}\left(\operatorname{sym}\nabla \boldsymbol{u_c} + iX_\chi \boldsymbol{c}\right) : \overline{\operatorname{sym}\nabla \boldsymbol{v}} = 0, & \forall \boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3), \\ \int_Y \boldsymbol{u_c} = 0. \end{cases}$$

$$\mathcal{A}^{\text{hom}}_\chi = (iX_\chi)^* \, \mathbb{A}^{\text{hom}} (iX_\chi),$$

Smoothing and averaging operators

Definition

The averaging operator on Y, $P_0 : L^2(Y;\mathbb{C}^3) \to \mathbb{C}^3 \hookrightarrow L^2(Y;\mathbb{C}^3)$ is given by

$$P_0\boldsymbol{u} = \int_Y \boldsymbol{u}.$$

That is, $P_0 = P_{\mathbb{C}^3}$, the orthogonal projection of $L^2(Y;\mathbb{C}^3)$ onto $\mathbb{C}^3$.

$$\mathbb{C}^3 = \mathrm{Eig}\,(0;\mathcal{A}_0) = \ker(\mathcal{A}_0) = \ker(\mathrm{sym}\,\nabla) \cap H^1_\#(Y;\mathbb{C}^3).$$

Definition

For $\varepsilon > 0$, the smoothing operator $\Xi_\varepsilon : L^2(\mathbb{R}^3;\mathbb{C}^3) \to L^2(\mathbb{R}^3;\mathbb{C}^3)$ is defined as follows:

$$\Xi_\varepsilon\boldsymbol{u} := \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} P_0 d\chi \right) \mathcal{G}_\varepsilon \boldsymbol{u} = \mathcal{G}_\varepsilon^* \left(\int_Y (\mathcal{G}_\varepsilon \boldsymbol{u})(y,\cdot) dy \right)$$

$$\left(\Xi_\varepsilon \boldsymbol{f}\right)(x) = \left(\mathcal{F}^{-1}(\mathbb{1}_{(2\pi\varepsilon)^{-1}Y'}) * \boldsymbol{f}\right)(x).$$

Connection with the full-space operators

Proposition (Passing to the unit cell for $\mathcal{A}_\varepsilon$)

The following identities hold for $z \in \rho(\mathcal{A}_\varepsilon)$

$$\mathcal{A}_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \frac{1}{\varepsilon^2} \mathcal{A}_\chi d\chi \right) \mathcal{G}_\varepsilon, \quad (\mathcal{A}_\varepsilon - zI)^{-1} = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi - zI \right)^{-1} d\chi \right) \mathcal{G}_\varepsilon$$

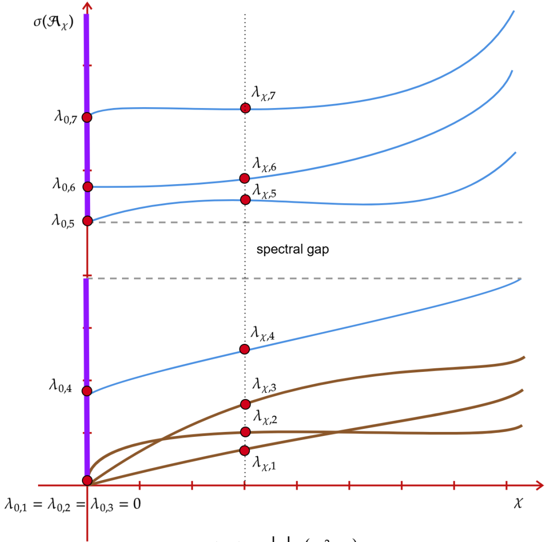
Proposition (Passing to the unit cell for $\mathcal{A}^{\text{hom}}$)

The following identities hold for $z \in \rho(\mathcal{A}^{\text{hom}})$

$$\mathcal{A}^{\text{hom}} \Xi_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \frac{1}{\varepsilon^2} P_0^* \mathcal{A}_\chi^{\text{hom}} P_0 \, d\chi \right) \mathcal{G}_\varepsilon.$$

$$\left(\mathcal{A}^{\text{hom}} - zI \right)^{-1} \Xi_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} - zI_{\mathbb{C}^3} \right)^{-1} P_0 d\chi \right) \mathcal{G}_\varepsilon.$$

Illustration, spectral decomposition of $\mathcal{A}_\varepsilon$



$$\sigma(\mathcal{A}_\varepsilon) = \bigcup_{\chi \in Y'} \sigma\left(\varepsilon^{-2}\mathcal{A}_\chi\right)$$

Fiber-wise spectral analysis

the spectrum of $\mathcal{A}_\chi$ is discrete, with eigenvalues λ_n^χ

$$0 \leqslant \lambda_1^\chi \leqslant \lambda_2^\chi \leqslant \lambda_3^\chi \leqslant \lambda_4^\chi \leqslant \cdots \to \infty$$

Lemma

There exists a closed contour $\Gamma \subset \{z \in \mathbb{C}, \Re(z) > 0\}$, oriented anticlockwise, where the following are valid:

- ***(Separation of spectrum)*** *There exist some $\mu > 0$, such that for each $\chi \in [-\mu, \mu]^3 \setminus \{0\}$, Γ encloses the three smallest eigenvalues of the operators $\frac{1}{|\chi|^2}\mathcal{A}_\chi$ and $\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}$. That is, the points*

$$|\chi|^{-2}\lambda_i^\chi, \quad |\chi|^{-2}\lambda_i^{\textit{hom},\chi}, \quad i = 1, 2, 3.$$

  Furthermore, Γ does not enclose any other eigenvalues

- ***(Buffer between contour and spectra)*** *There exist some $\rho_0 > 0$ such that*

$$\inf_{\substack{z\in\Gamma,\\ \chi\in[-\mu,\mu]^3\setminus\{0\}\\ i\in\{1,2,3,4\}}} \left| z - \frac{1}{|\chi|^2}\lambda_i^\chi \right| \geqslant \rho_0 \qquad \textit{and} \qquad \inf_{\substack{z\in\Gamma,\\ \chi\in[-\mu,\mu]^3\setminus\{0\}\\ i\in\{1,2,3\}}} \left| z - \frac{1}{|\chi|^2}\lambda_i^{\textit{hom},\chi} \right| \geqslant \rho_0.$$

Illustration

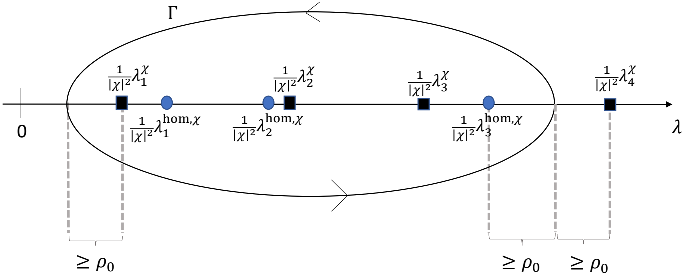


Figure: A schematic of the contour Γ, for quasimomentum $\chi \in [-\mu, \mu]^3 \backslash \{0\}$.

Definition

Let $P_\chi : L^2(Y; \mathbb{C}^3) \to L^2(Y; \mathbb{C}^3)$ be the projection onto the eigenspace corresponding to the first three eigenvalues λ_1^χ, λ_2^χ, λ_3^χ of $\mathcal{A}_\chi$.

Heuristic

Scaled resolvent problem

Find $\boldsymbol{u} \in H^1_\#(Y;\mathbb{C}^3)$ such that

$$\frac{1}{|\chi|^2}\int_Y \mathbb{A}\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u} : \overline{\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{v}} - z\int_Y \boldsymbol{u}\cdot\overline{\boldsymbol{v}} = \int_Y \boldsymbol{f}\cdot\overline{\boldsymbol{v}}, \quad \forall\, \boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3).$$

Our goal is to expand the solution $\boldsymbol{u}$ in the form

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_{\text{err}}, \quad \boldsymbol{u}_i,\, \boldsymbol{u}_{\text{err}} \in H^1_\#(Y;\mathbb{C}^3), \quad i = 0,1,2,$$

where the terms satisfy the following bounds

$$\boldsymbol{u}_0 = \mathcal{O}(1), \quad \boldsymbol{u}_1 = \mathcal{O}(|\chi|), \quad \boldsymbol{u}_2 = \mathcal{O}(|\chi|^2), \quad \text{as } |\chi| \downarrow 0.$$

with respect to $H^1(Y;\mathbb{C}^3)$ norm, with explicit dependence on $\|\boldsymbol{f}\|_{L^2(Y;\mathbb{C}^3)}$, $z \in \mathbb{C}$.

Engineered equations

Leading order resolvent problem: Find $u_0 \in \mathbb{C}^3$ such that:

$$\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}}\boldsymbol{u}_0 - z\boldsymbol{u}_0 = \int_Y \boldsymbol{f}.$$

The first order corrector: Find $u_1 \in H^1_\#(Y;\mathbb{C}^3)$, $\int_Y u_1 = 0$ such that:

$$\int_Y \mathbb{A}\,\text{sym}\,\nabla\boldsymbol{u}_1 : \overline{\text{sym}\,\nabla\boldsymbol{v}} = -\int_Y \mathbb{A}iX_\chi\boldsymbol{u}_0 : \overline{\text{sym}\,\nabla\boldsymbol{v}}, \quad \forall\boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3).$$

The second order corrector: Find $u_2 \in H^1_\#(Y;\mathbb{C}^3)$, $\int_Y u_2 = 0$ such that:

$$\int_Y \mathbb{A}\,\text{sym}\,\nabla\boldsymbol{u}_2 : \overline{\text{sym}\,\nabla\boldsymbol{v}} = -\int_Y \mathbb{A}iX_\chi\boldsymbol{u}_1 : \overline{\text{sym}\,\nabla\boldsymbol{v}} - \int_Y \mathbb{A}\,\text{sym}\,\nabla\boldsymbol{u}_1 : \overline{iX_\chi\boldsymbol{v}}$$
$$-\int_Y \mathbb{A}iX_\chi\boldsymbol{u}_0 : \overline{iX_\chi\boldsymbol{v}} + z|\chi|^2\int_Y \boldsymbol{u}_0\cdot\overline{\boldsymbol{v}} + |\chi|^2\int_Y \boldsymbol{f}\cdot\overline{\boldsymbol{v}}, \quad \forall\boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3).$$

Compute the estimates

- Equation for the error term

$$\frac{1}{|\chi|^2}\int_Y \mathbb{A}\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{u}_{\text{err}} : \overline{\left(\operatorname{sym}\nabla + iX_\chi\right)\boldsymbol{v}} - z\int_Y \boldsymbol{u}_{\text{err}}\cdot\overline{\boldsymbol{v}} = \frac{1}{|\chi|^2}\mathcal{R}_{\text{err}}(\boldsymbol{v}),$$

$\forall \boldsymbol{v} \in H^1_\#(Y;\mathbb{C}^3)$,

- Operator norm estimate for the residual

$$\left\| \frac{1}{|\chi|^2}\mathcal{R}_{\mathsf{err}} \right\|_{(H^1(Y;\mathbb{C}^3))^*} \leqslant C\left[\frac{\max\{1,|z|^2\}}{D_{\text{hom}}(z)} + \max\{1,|z|\}\right]|\chi|\|\boldsymbol{f}\|_{L^2}.$$

Theorem

Let $\chi \in Y'\backslash\{0\}$ and $z \in \rho(\frac{1}{|\chi|^2}\mathcal{A}_\chi) \cap \rho(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{hom})$. There exists a constant $C > 0$, which does not depend on χ and z, such that the following norm-resolvent estimate holds:

$$\left\| \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - zI\right)^{-1} - \left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - zI_{\mathbb{C}^3}\right)^{-1} P_0 \right\|_{L^2(Y;\mathbb{C}^3)\to H^1(Y;\mathbb{C}^3)}$$

$$\leqslant C\max\left\{1, \frac{|z+1|}{D(z)}\right\}\left[\frac{\max\{1,|z|^2\}}{D_{hom}(z)} + \max\{1,|z|\}\right]|\chi|,$$

Separation of spectrum

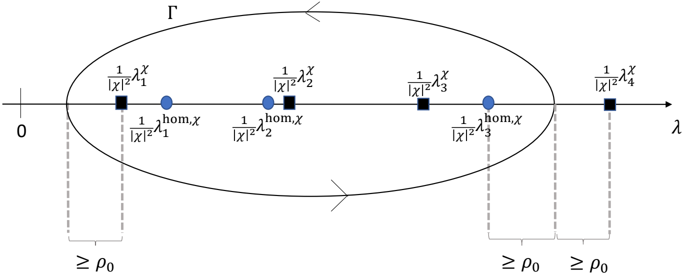


Figure: A schematic of the contour Γ, for quasimomentum $\chi \in [-\mu, \mu]^3 \backslash \{0\}$.

$g_{\varepsilon,\chi} : \{z \in \mathbb{C} : \Re(z) > 0\} \to \mathbb{C}$

$$g_{\varepsilon,\chi}(z) := \left(\frac{|\chi|^2}{\varepsilon^2} z + 1 \right)^{-1}.$$

Go back to the correct scaling

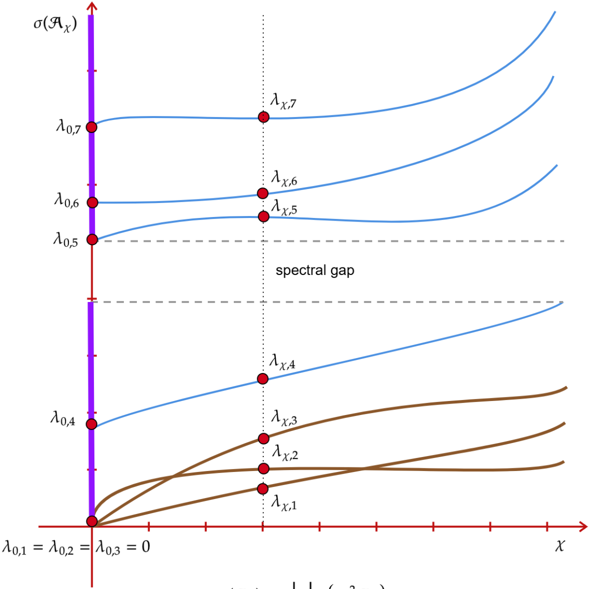
Step 1: Estimates on $L^2(Y;\mathbb{C}^3)$. For each $\chi \in Y'$, decompose the resolvent of $\frac{1}{\varepsilon^2}\mathcal{A}_\chi$ as follows:

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1} = P_\chi\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1} P_\chi + (I - P_\chi)\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1}(I - P_\chi).$$

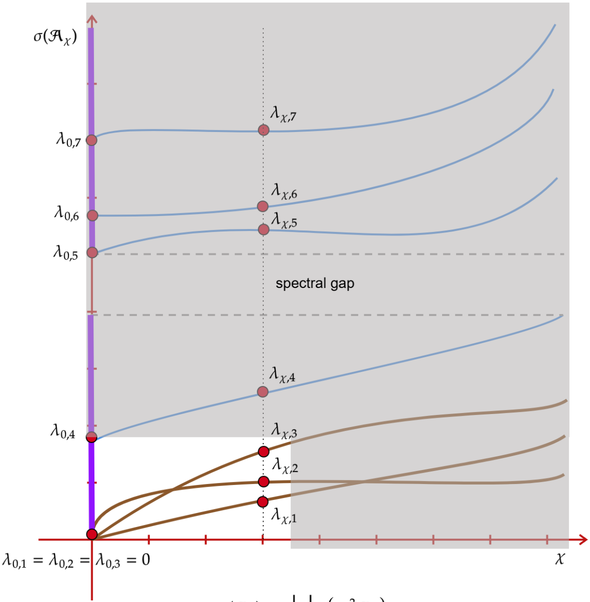
Now fix $\chi \in [-\mu,\mu]^3\backslash\{0\}$. By the Cauchy integral formula with contour Γ,

$$P_\chi\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1} P_\chi = -\frac{1}{2\pi i}\oint_\Gamma g_{\varepsilon,\chi}(z)\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi - zI\right)^{-1} dz,$$

$$P_0\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3}\right)^{-1} P_0 = -\frac{1}{2\pi i}\oint_\Gamma g_{\varepsilon,\chi}(z)\left(\frac{1}{|\chi|^2}\mathcal{A}_\chi^{\text{hom}} - zI\right)^{-1} dz.$$



$$\sigma(\mathcal{A}_\varepsilon) = \bigcup_{\chi \in Y'} \sigma\left(\varepsilon^{-2}\mathcal{A}_\chi\right)$$



$$\sigma(\mathcal{A}_\varepsilon) = \bigcup_{\chi \in Y'} \sigma\left(\varepsilon^{-2}\mathcal{A}_\chi\right)$$

Norm resolvent estimates

Now, **for small** χ and **on the bottom of the spectrum of** $\mathcal{A}_\chi$, we have:

$$\begin{aligned}
&\left\| P_\chi \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} P_\chi - \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3} \right)^{-1} P_0 \right\|_{L^2 \to L^2} \\
&\leqslant \frac{1}{2\pi} \oint_\Gamma |g_{\varepsilon,\chi}(z)| \left\| \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi - zI \right)^{-1} - \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi^{\text{hom}} - zI_{\mathbb{C}^3} \right)^{-1} P_0 \right\|_{L^2 \to L^2} dz \\
&\leqslant C \left(\max\left\{ \frac{|\chi|^2}{\varepsilon^2}, 1 \right\} \right)^{-1} |\chi| \\
&\leqslant C\varepsilon.
\end{aligned}$$

But what about:

- Large χ?
- The bulk of the spectrum? $(I - P_\chi)\left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi + I\right)^{-1}(I - P_\chi)$
- Getting rid of the smoothing operator Ξ_ε? Recall $\left(\mathcal{A}^{\text{hom}} - zI\right)^{-1} \Xi_\varepsilon = \mathcal{G}_\varepsilon^* \left(\int_{Y'}^{\oplus} \left(\frac{1}{\varepsilon^2}\mathcal{A}_\chi^{\text{hom}} - zI_{\mathbb{C}^3}\right)^{-1} P_0 d\chi \right) \mathcal{G}_\varepsilon.$

Norm resolvent estimates

Now, **for small** χ and **on the bottom of the spectrum of** $\mathcal{A}_\chi$, we have:

$$\left\| P_\chi \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi + I \right)^{-1} P_\chi - \left(\frac{1}{\varepsilon^2} \mathcal{A}_\chi^{\text{hom}} + I_{\mathbb{C}^3} \right)^{-1} P_0 \right\|_{L^2 \to L^2}$$

$$\leqslant \frac{1}{2\pi} \oint_\Gamma |g_{\varepsilon,\chi}(z)| \left\| \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi - zI \right)^{-1} - \left(\frac{1}{|\chi|^2} \mathcal{A}_\chi^{\text{hom}} - zI_{\mathbb{C}^3} \right)^{-1} P_0 \right\|_{L^2 \to L^2} dz$$

$$\leqslant C \left(\max \left\{ \frac{|\chi|^2}{\varepsilon^2}, 1 \right\} \right)^{-1} |\chi|$$

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- Large χ?
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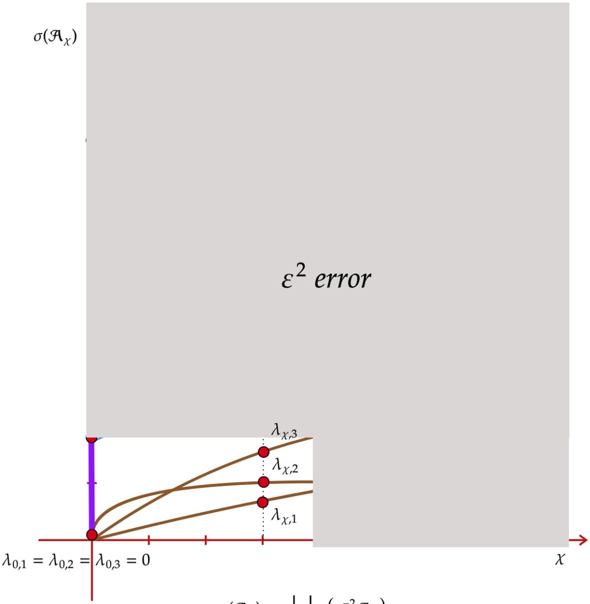
It all produces an error of order ε^2!



$$\sigma(\mathcal{A}_\varepsilon) = \bigcup_{\chi \in Y'} \sigma\left(\varepsilon^{-2}\mathcal{A}_\chi\right)$$

The end

Thank you for attention!