Homogenisation of fractional diffusion problems

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Stationary diffusion equation



$\Omega \subseteq \mathbf{R}^d$ open and bounded, $f \in \mathrm{H}^{-1}(\Omega)$, $\mathbf{A} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_d(\mathbf{R}))$ given; stationary diffusion equation with homogenous Dirichlet boundary condition:

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}\nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

Physical interpretations:

- Thermal conductivity; $\mathbf{A}$ – (thermal) conductivity of material, f – external heat density, u – temperature
- Electrical conductivity; $\mathbf{A}$ – (electrical) conductivity of material, f – electric charge density, u – electrical potential

Homogenisation



The physical idea of homogenisation - averaging heterogeneous media in order to derive effective properties

$$\begin{cases} -\text{div}\,(\mathbf{A}_n \nabla u_n) = f \\ u_n \in \mathrm{H}_0^1(\Omega) \end{cases}$$

If $u_n \to u$, $\mathbf{A}_n \to \mathbf{A}$, homogenized problem is

$$\begin{cases} -\text{div}\,(\mathbf{A} \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

What are the right topologies?

H -convergence (Murat&Tartar, 1970s)



For $0 < \alpha < \beta$, $\Omega \subseteq \mathbf{R}^d$ open and bounded, we define the set of admissible coefficients:

$$\mathcal{M}(\alpha, \beta; \Omega) := \left\{ \mathbf{A} \in L^\infty(\Omega; \mathbf{R}^{d\times d}) : \mathbf{A}(x)\xi \cdot \xi \geq \alpha|\xi|^2, \mathbf{A}^{-1}(x)\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \right\}.$$

Definition (H-convergence)

Let $(\mathbf{A}_n)_n$, $\mathbf{A}$ in $\mathcal{M}(\alpha, \beta; \Omega)$. Then $(\mathbf{A}_n)_n$ is said to H-**converge** to $\mathbf{A}$ ($\mathbf{A}_n \xrightarrow{H} \mathbf{A}$), if for all $f \in \mathrm{H}^{-1}(\Omega)$ the sequence u_n of solutions of

$$\begin{cases} -\mathrm{div}\,(\mathbf{A}_n \nabla u_n) = f \\ u_n \in \mathrm{H}_0^1(\Omega) \end{cases}$$

satisfies $u_n \xrightarrow{\mathrm{H}_0^1(\Omega)} u$ and $\mathbf{A}_n \nabla u_n \xrightarrow{\mathrm{L}^2(\Omega)^d} \mathbf{A}\nabla u$, where

$$\begin{cases} -\mathrm{div}\,(\mathbf{A} \nabla u) = f \\ u \in \mathrm{H}_0^1(\Omega) \end{cases}$$

$(\mathcal{M}(\alpha, \beta; \Omega), \tau_H)$ **is compact and metrisable.**

Fractional diffusion problem

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\,\nabla^{\mathrm{s}} u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbf{R}^d \setminus \Omega. \end{cases}$$

Nonlocal operators ($s \in \langle 0, 1\rangle$):

- ∇^{s}, $\operatorname{div}^{\mathrm{s}}$ - Riesz fractional gradient and divergence:

$$\nabla^{\mathrm{s}}\varphi(x) = \mu_s \int_{\mathbf{R}^d} \frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|^{d+s+1}}\,dy,\ \operatorname{div}^{\mathrm{s}}\psi(x) = \mu_s \int_{\mathbf{R}^d} \frac{(\psi(x)-\psi(y))\cdot(x-y)}{|x-y|^{d+s+1}}\,dy,$$

$\mu_s = 2^s \pi^{-\frac{d}{2}} \frac{\Gamma((d+s+1)/2)}{\Gamma((1-s)/2)}$.

M.Shílhavý. Fractional vector analysis based on invariance requirements (critique of coordinate approaches), Continuum Mech. Thermodyn. 32 (2019) 207-228.

T. Shieh and D. Spector. On a new class of fractional partial differential equations, Adv. Calc. Var 8 (2015) 321-336.

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\,\nabla^{\mathrm{s}}u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbf{R}^d \setminus \Omega. \end{cases} \tag{1}$$

- For $s \in \langle 0, 1\rangle$,

$$\|u\|_s := \left(\|u\|^2_{\mathrm{L}^2(\mathbf{R}^d)} + \|\nabla^{\mathrm{s}}u\|^2_{\mathrm{L}^2(\mathbf{R}^d)}\right)^{1/2}, \quad \mathrm{H}^s(\mathbf{R}^d) = \mathsf{Cl}_{\|\cdot\|_s}\mathrm{C}^\infty_c(\mathbf{R}^d),$$

- For $\Omega \subseteq \mathbf{R}^d$ open

$$\mathrm{H}^s_0(\Omega) := \mathsf{Cl}_{\|\cdot\|_s}\mathrm{C}^\infty_c(\Omega) \subseteq \mathrm{H}^s(\mathbf{R}^d), \quad \mathrm{H}^{-s}(\Omega) := (\mathrm{H}^s_0(\Omega))' .$$

- $u \in \mathrm{H}^s_0(\Omega)$, $f \in \mathrm{H}^{-s}(\Omega)$, $\mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d)$ $u \in \mathrm{H}^s_0(\Omega)$ is a **weak** or **variational solution of the problem** (1) if for any $v \in \mathrm{H}^s_0(\Omega)$

$$\int_{\mathbf{R}^d} \mathbf{A}(x)\,\nabla^{\mathrm{s}}u(x) \cdot \nabla^{\mathrm{s}}v(x)\,dx = {}_{\mathrm{H}^{-s}(\Omega)}\langle f, v\rangle_{\mathrm{H}^s_0(\Omega)} .$$



Definition (H^s - convergence)

Let $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d)$. Then $\mathbf{A}_n$ is said to H^s -converges to $\mathbf{A}$ if for every $f \in \mathrm{H}^{-s}(\Omega)$, the solutions $u_n \in \mathrm{H}_0^s(\Omega)$ of $-\operatorname{div}^{\mathrm{s}}(A_n \nabla^{\mathrm{s}} u_n) = f$ satisfy

$$u_n \xrightarrow{\mathrm{H}_0^s(\Omega)} u \qquad \mathbf{A}_n \nabla^{\mathrm{s}} u_n \xrightarrow{\mathrm{L}^2(\mathbf{R}^d)^d} \mathbf{A} \nabla^{\mathrm{s}} u,$$

where $u \in \mathrm{H}_0^s(\Omega)$ solves $-\operatorname{div}^{\mathrm{s}}(\mathbf{A} \nabla^{\mathrm{s}} u) = f$.


M. Caponi, A. Carbotti, and A. Maione. H-compactness for nonlocal linear operators in fractional divergence form, Calculus of Variations and Partial Differential Equations 64/290 (2025)

H^s -convergence: characterisation



Caponi et all (2025): fix $\mathbf{A}_0 \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d)$ and define

$$\mathcal{M}(\alpha, \beta; \Omega, \mathbf{A}_0) := \{\mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d) : \mathbf{A}|_{\mathbf{R}^d \setminus \Omega} = \mathbf{A}_0|_{\mathbf{R}^d \setminus \Omega}\}.$$

For $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \Omega, \mathbf{A}_0)$, $\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \iff \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega$.

- Although two notions of convergence are equivalent (in the above sense), the corresponding diffusion problems differ significantly due to their local/nonlocal nature, which influences both the solution and the diffusion process.

H^s -convergence: characterisation

Our contribution: for $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d)$

$$\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \text{ and } \mathbf{A}_n|_{\mathbf{R}^d \setminus \overline{\Omega}} \overset{*}{\rightharpoonup} \mathbf{A}|_{\mathbf{R}^d \setminus \overline{\Omega}}$$

Consequences:

- compactness of H^s -convergence on $\mathcal{M}(\alpha, \beta; \mathbf{R}^d)$
- metrisability, there exist a metric $d : \mathcal{M}(\alpha, \beta; \mathbf{R}^d) \to [0, \infty\rangle$,

$$d(\mathbf{A}, \mathbf{B}) = d_H(\mathbf{A}|_\Omega, \mathbf{B}|_\Omega) + d_*(\mathbf{A}|_{\mathbf{R}^d \setminus \mathrm{Cl}\,\Omega}, \mathbf{B}|_{\mathbf{R}^d \setminus \mathrm{Cl}\,\Omega})$$

- transpose invariance of H^s -limit, energy convergence, locality property, the equivalence between H^s - and G^s -convergence
- H^s -convergence is a special case of nonlocal H - convergence when restricted to local (multiplicative) operators

A. Buchinger, K. Burazin, I. C., M. Erceg, M. Jolić, M. Waurick, Characterisation of homogenisation for nonlocal diffusion by local topologies, arXiv:2601.17579 (2026)

H^s -convergence: characterisation



Our contribution: for $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^d)$

$$\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \text{ and } \mathbf{A}_n|_{\mathbf{R}^d \setminus \overline{\Omega}} \overset{*}{\rightharpoonup} \mathbf{A}|_{\mathbf{R}^d \setminus \overline{\Omega}}$$

Key ideas in the proof:

- **(Continuity of the Riesz potential)** $I_{1-s} : \mathrm{H}_0^s(\Omega) \to \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R}^d)$ is linear and continuous, and $\nabla(I_{1-s}u) = \nabla^{\mathrm{s}} u$ a.e. in Ω.
- **(Poincaré inequality)** $\|u\|_{\mathrm{L}^2(\Omega)} \lesssim_{d,s,\Omega} \|\nabla^{\mathrm{s}} u\|_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)}$
- **(Rellich–Kondrašov theorem)** $\mathrm{H}_0^s(\Omega)$ is compactly embedded into $\mathrm{L}^2(\Omega)$
- **(Compactness on the complement)** For $\Omega \Subset \Omega_1$ the mapping $\mathrm{H}_0^s(\Omega) \ni u \mapsto \nabla^{\mathrm{s}} u|_{\Omega_1^c} \in \mathrm{L}^2(\Omega_1^c)$ is compact

Why truncate? The δ-horizon operator



- The Riesz fractional gradient ∇^{s} is **fully nonlocal**
- For applications one truncates the interaction to a finite **horizon** $\delta > 0$, using a radial cut-off $w_\delta \in \mathrm{C}_c^\infty(B(0,\delta))$, $w_\delta \equiv 1$ near 0:

$$\mathrm{D}_\delta^{\mathrm{s}}\,\varphi(x) := \mu_s \int_{\mathbf{R}^d} \frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|^{d+s+1}}\, w_\delta(x-y)\,dy,$$

$$\mathrm{div}_\delta^{\mathrm{s}}\,\psi(x) := \mu_s \int_{\mathbf{R}^d} \frac{(\psi(x)-\psi(y))\cdot(x-y)}{|x-y|^{d+s+1}}\, w_\delta(x-y)\,dy.$$

- If φ is supported in Ω, then $\mathrm{D}_\delta^{\mathrm{s}}\,\varphi$ is supported in $\Omega^\delta := \Omega + B(0,\delta)$ (the δ-**closure**) – *not* on all of $\mathbf{R}^d$


J. Cueto, C. Kreisbeck, H. Schönberger. A variational theory for integral functionals involving finite-horizon fractional gradients, Fract. Calc. Appl. Anal. 26 (2023) 2001–2056.

δ-horizon fractional diffusion problem

$$\begin{cases} -\operatorname{div}_\delta^{\mathrm{s}}(\mathbf{A}\,\mathrm{D}_\delta^{\mathrm{s}}\,u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbf{R}^d \setminus \Omega. \end{cases} \tag{2}$$

- For $s \in \langle 0, 1\rangle$,

$$\|u\|_{s,\delta} := \left(\|u\|^2_{\mathrm{L}^2(\mathbf{R}^d)} + \|\,\mathrm{D}_\delta^{\mathrm{s}}\,u\|^2_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)}\right)^{1/2}$$

$$\mathrm{H}^{s,\delta}(\mathbf{R}^d) = \mathsf{Cl}_{\|\cdot\|_{s,\delta}}\mathrm{C}_c^\infty(\mathbf{R}^d) = \mathrm{H}^s(\mathbf{R}^d),$$

- For $\Omega \subseteq \mathbf{R}^d$ open

$$\mathrm{H}_0^{s,\delta}(\Omega) := \mathsf{Cl}_{\|\cdot\|_{s,\delta}}\mathrm{C}_c^\infty(\Omega) = \mathrm{H}_0^s(\Omega), \quad \mathrm{H}^{-s,\delta}(\Omega) := \left(\mathrm{H}_0^{s,\delta}(\Omega)\right)'.$$

- $u \in \mathrm{H}_0^{s,\delta}(\Omega)$, $f \in \mathrm{H}^{-s,\delta}(\Omega)$, $\mathbf{A} \in \mathcal{M}(\alpha, \beta; \Omega_\delta)$
- $u \in \mathrm{H}_0^{s,\delta}(\Omega)$ is a **weak** or **variational solution of the problem** (2) if for any $v \in \mathrm{H}_0^{s,\delta}(\Omega)$

$$\int_{\Omega_\delta} \mathbf{A}(x)\,\mathrm{D}_\delta^{\mathrm{s}}\,u(x) \cdot \mathrm{D}_\delta^{\mathrm{s}}\,v(x)\,dx = {}_{\mathrm{H}^{-s,\delta}(\Omega)}\langle f, v\rangle_{\mathrm{H}_0^{s,\delta}(\Omega)}.$$

Definition ($H^{s,\delta}$ - convergence)

Let $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \Omega_\delta)$. Then $\mathbf{A}_n$ is said to $H^{s,\delta}$ -converges to $\mathbf{A}$ if for every $f \in \mathrm{H}^{-s,\delta}(\Omega)$, the solutions $u_n \in \mathrm{H}_0^s(\Omega)$ of $-\operatorname{div}_\delta^{\mathrm{s}}(A_n \, \mathrm{D}_\delta^{\mathrm{s}} u_n) = f$ satisfy

$$u_n \xrightarrow{\mathrm{H}_0^{s,\delta}(\Omega)} u \qquad \mathbf{A}_n \, \mathrm{D}_\delta^{\mathrm{s}} u_n \xrightarrow{\mathrm{L}^2(\Omega_\delta)^d} \mathbf{A} \, \mathrm{D}_\delta^{\mathrm{s}} u,$$

where $u \in \mathrm{H}_0^s(\Omega)$ solves $-\operatorname{div}_\delta^{\mathrm{s}}(\mathbf{A} \, \mathrm{D}_\delta^{\mathrm{s}} u) = f$.

For $(\mathbf{A}_n)_n, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \Omega_\delta)$

$$\boxed{\mathbf{A}_n \xrightarrow{H^{s,\delta}} \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \text{ and } \mathbf{A}_n|_{\Omega_\delta \setminus \overline{\Omega}} \overset{*}{\rightharpoonup} \mathbf{A}|_{\Omega_\delta \setminus \overline{\Omega}}}$$

- compactness of $H^{s,\delta}$-convergence on $\mathcal{M}(\alpha, \beta; \Omega_\delta)$, uniqueness of the limit, metrisability, locality and transpose invariance, energy convergence, and equivalence with $G^{s,\delta}$-convergence on symmetric coefficients.

Limiting regime



- As $\delta \to \infty$, the truncated operators converge to the Riesz fractional ones:

$$\| \mathrm{D}^{\mathrm{s}}_{\delta} - \nabla^{\mathrm{s}} \|_{\mathcal{L}(\mathrm{H}^{s}(\mathbf{R}^{d}),\mathrm{L}^{2}(\mathbf{R}^{d}))} \to 0$$

- if $\mathbf{A}_{\delta}, \mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbf{R}^{d})$ such that $\mathbf{A}_{\delta} \xrightarrow{H^{s}} \mathbf{A}$ as $\delta \to \infty$, then the δ-horizon solutions of

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}_{\delta}(\mathbf{A}_{\delta}\, \mathrm{D}^{\mathrm{s}}_{\delta}\, u_{\delta}) = f \quad \text{in } \Omega, \\ u_{\delta} \in \mathrm{H}^{s}_{0}(\Omega) \end{cases}$$

converges to the weak solution of the fractional diffusion problem

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\, \nabla^{\mathrm{s}} u) = f \quad \text{in } \Omega, \\ u \in \mathrm{H}^{s}_{0}(\Omega), \end{cases}$$

i.e.

$$u_{\delta} \xrightarrow{\mathrm{H}^{s}_{0}(\Omega)} u, \qquad \mathbf{A}_{\delta}\, \mathrm{D}^{\mathrm{s}}_{\delta}\, u_{\delta} \xrightarrow{\mathrm{L}^{2}(\mathbf{R}^{d})} \mathbf{A}\, \nabla^{\mathrm{s}} u.$$

Relation between H^s - and $H^{s,\delta}$ -convergence



- For a sequence $(\mathbf{A}_n)_n$ in $\mathcal{M}(\alpha,\beta;\mathbf{R}^d)$, and $\mathbf{A}\in\mathcal{M}(\alpha,\beta;\mathbf{R}^d)$, the following holds

$$\begin{aligned}
\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} &\iff \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \quad\text{and}\quad \mathbf{A}_n|_{\mathbf{R}^d\setminus \mathsf{Cl}\,\Omega} \xrightarrow{*} \mathbf{A}|_{\mathbf{R}^d\setminus \mathsf{Cl}\,\Omega}\\
&\iff \forall\delta>0\quad \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \quad\text{and}\quad \mathbf{A}_n|_{\Omega_\delta\setminus \mathsf{Cl}\,\Omega} \xrightarrow{*} \mathbf{A}|_{\Omega_\delta\setminus \mathsf{Cl}\,\Omega}\\
&\iff \forall\delta>0\quad \mathbf{A}_n|_{\Omega_\delta} \xrightarrow{H^{s,\delta}} \mathbf{A}|_{\Omega_\delta}.
\end{aligned}$$