Homogenization theory of the Reissner-Mindlin plate equation

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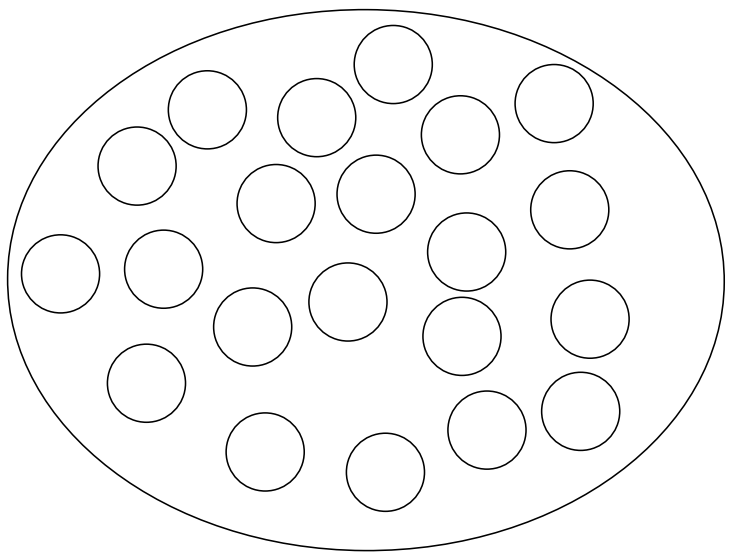
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1. Introduction
2. H-convergence
3. Properties of H-convergence
4. Corrector results



- The physical ideas of homogenization, i.e., averaging heterogeneous media in order to derive effective properties
- Mathematical approach by operator convergence introduced by Spagnolo through G- convergence and generalized by Murat and Tartar through H-convergence



Reissner-Mindlin plate equation



Homogeneous Dirichlet boundary value problem:

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}\nabla\vec{\varphi}) + S(\vec{\varphi} + \nabla w) = \vec{f} \text{ in } \Omega \\ -\mathrm{div}\,(S(\vec{\varphi} + \nabla w) = g \text{ in } \Omega \\ (\vec{\varphi}, w) \in [\mathrm{H}_0^1(\Omega))]^3 \end{cases}$$

Reissner-Mindlin plate equation



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- $\vec{\varphi} \in [\mathrm{H}_0^1(\Omega)]^2$ rigid rotation of the transversal material fiber
- $w \in \mathrm{H}_0^1(\Omega)$ transversal displacement
- $\Omega \subseteq \mathbf{R}^2$ bounded
- $\vec{f} \in [\mathrm{H}^{-1}(\Omega)]^2$, $g \in \mathrm{H}^{-1}(\Omega)$ external forces



- $\mathbf{P} \in \mathcal{T}(\Omega, \alpha_1, \beta_1)$, $0 < \alpha_1 < \beta_1$ where

$$\mathcal{T}(\Omega, \alpha_1, \beta_1) = \{T \in L^\infty(\Omega; \mathcal{L}(\mathcal{M}_2, \mathcal{M}_2)) : (\forall\, A \in \mathcal{M}_2)$$
$$T(\mathbf{x})A : A \geq \alpha_1 A : A,\ T^{-1}(\mathbf{x})A : A \geq \frac{1}{\beta_1} A : A,\ \text{a.e.}\, \mathbf{x} \in \Omega\}$$

- $S \in \mathcal{M}(\Omega, \alpha_2, \beta_2)$, $0 < \alpha_2 < \beta_2$ where

$$\mathcal{M}(\Omega, \alpha_2, \beta_2) = \{M \in L^\infty(\Omega; \mathcal{M}_2) : (\forall\, \xi \in \mathbf{R}^2)$$
$$M(\mathbf{x})\xi \cdot \xi \geq \alpha_2 \xi \cdot \xi,\ M^{-1}(\mathbf{x})\xi \cdot \xi \geq \frac{1}{\beta_2} \xi \cdot \xi,\ \text{a.e.}\, \mathbf{x} \in \Omega\},$$



Definition

A sequence $(\mathbf{P}_n, S_n) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ H- converges to $(\mathbf{P}^*, S^*) \in \mathcal{T}(\Omega, \alpha_1', \beta_1') \times \mathcal{M}(\Omega, \alpha_2', \beta_2')$ if for every $(\vec{f}, g) \in [\mathrm{H}^{-1}(\Omega)]^3$ a sequence of solutions $(\vec{\varphi}_n, w_n)$ of problems

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}_n \nabla \vec{\varphi}_n) + S_n(\vec{\varphi}_n + \nabla w_n) = \vec{f} \\ -\mathrm{div}\,(S_n(\vec{\varphi}_n + \nabla w_n) = g \\ (\vec{\varphi}_n, w_n) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}$$

converges weakly to a limit $(\vec{\varphi}, w) \in [\mathrm{H}_0^1(\Omega)]^3$ while the sequence $\mathbf{P}_n \nabla \vec{\varphi}_n$ converges weakly to $\mathbf{P}^* \nabla \vec{\varphi}$ in $\mathrm{L}^2(\Omega, \mathcal{M}_2)$ and $S_n(\vec{\varphi}_n + \nabla w_n)$ converges weakly to $S^*(\vec{\varphi} + \nabla w) \in [\mathrm{L}^2(\Omega)]^2$. In that case, $(\mathbf{P}^*, S^*)$ is called H-limit of the sequence $(\mathbf{P}_n, S_n)$. Also note that $(\vec{\varphi}, w)$ is a solution to

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}^* \nabla \vec{\varphi}) + S^*(\vec{\varphi} + \nabla w) = \vec{f} \text{ in } \Omega \\ -\mathrm{div}\,(S^*(\vec{\varphi} + \nabla w) = g \text{ in } \Omega \\ (\vec{\varphi}, w) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases} .$$



Theorem (Compactness theorem)

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$. *Then there is a subsequence* $(\mathbf{P}_{n_k}, S_{n_k})$ *and*

$$(\mathbf{P}^*, S^*) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$$

such that

$$(\mathbf{P}_{n_k}, S_{n_k}) \text{ H-converges to } (\mathbf{P}^*, S^*).$$



Lemma (Div-rot lemma)

Let u_n and v_n be two sequences in $[\mathrm{L}^2(\Omega)]^d$ such that

$$u_n \rightharpoonup u \quad \text{in } \mathrm{L}^2(\Omega)^{\mathrm{d}}$$
$$v_n \rightharpoonup v \quad \text{in } \mathrm{L}^2(\Omega)^{\mathrm{d}}.$$

Assume furthermore that

$$\operatorname{div} u_n \longrightarrow \operatorname{div} u \quad \text{in } \mathrm{H}^{-1}(\Omega)$$
$$\operatorname{rot} v_n \longrightarrow \operatorname{rot} v \quad \text{in } \mathrm{H}^{-1}(\Omega; \mathcal{M}_{\mathrm{d}}).$$

Then $u_n \cdot v_n \rightharpoonup u \cdot v$ in the sense of distributions.



Theorem (Oscillating test functions)

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *which* H*-converges to* $(\mathbf{P}^*, S^*)$*. For* $i, j = 1, 2$ *there exist* $\vec{\psi}_n^{ij} \in [\mathrm{H}^1(\Omega)]^2$ *and* $v_n^i \in \mathrm{H}^1(\Omega)$ *such that*

$$
\begin{aligned}
\vec{\psi}_n^{ij} &\longrightarrow x_i e_j \quad \text{in} \quad [\mathrm{H}^1(\Omega)]^2\,, \\
v_n^j &\longrightarrow x_j \quad \text{in} \quad \mathrm{H}^1(\Omega)\,, \\
F_n^{ij} = -\mathsf{div}\,(\mathbf{P}^n \nabla \vec{\psi}_n^{ij}) &\longrightarrow F^{ij} \quad \text{in} \quad [\mathrm{H}^{-1}(\Omega)]^2\,, \\
G_n^i = -\mathsf{div}\,(S_n \nabla v_n^i) &\longrightarrow G^i \quad \text{in} \quad \mathrm{H}^{-1}(\Omega)\,,
\end{aligned}
$$

where F^{ij} *and* G^i *are some elements of* $[\mathrm{H}^{-1}(\Omega)]^2$ *and* $\mathrm{H}^{-1}(\Omega)$*, respectively.*



Theorem

Let $F = \{(\vec{f}_n, g_n) : n \in \mathbf{N}\}$ be a dense countable family in $[\mathrm{H}^{-1}(\Omega)]^3$, $\mathbf{P}, \mathbf{Q}$ tensors in $\mathcal{T}(\Omega, \alpha_1, \beta_1)$, T, S matrices in $\mathcal{M}(\Omega, \alpha_2, \beta_2)$ and $(\vec{\varphi}_n, w_n)$, $(\vec{\psi}_n, v_n)$ sequences of solutions to

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}\nabla\vec{\varphi}_n) + S(\vec{\varphi}_n + \nabla w_n) = \vec{f}_n \\ -\mathrm{div}\,(S(\vec{\varphi}_n + \nabla w_n) = g_n \\ (\vec{\varphi}_n, w_n) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}$$

and

$$\begin{cases} -\mathrm{div}\,(\mathbf{Q}\nabla\vec{\psi}_n) + T(\vec{\psi}_n + \nabla v_n) = \vec{f}_n \\ -\mathrm{div}\,(T(\vec{\psi}_n + \nabla v_n) = g_n \\ (\vec{\psi}_n, v_n) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases},$$

respectively. Then

$$d((\mathbf{P}, S), (\mathbf{Q}, T)) = \sum_{n=1}^{\infty} 2^{-n} \left(\frac{\|\vec{\varphi}_n - \vec{\psi}_n\|_{[\mathrm{L}^2(\Omega)]^2} + \|w_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{P}\nabla\vec{\varphi}_n - \mathbf{Q}\nabla\vec{\psi}_n\|_{\mathrm{H}^{-1}(\Omega;\mathcal{M}_2)}}{\|(\vec{f}_n, g_n)\|_{[\mathrm{H}^{-1}(\Omega)]^3}} + \frac{\|S(\vec{\varphi}_n + \nabla w_n) - T(\vec{\psi}_n + \nabla v_n)\|_{[\mathrm{H}^{-1}(\Omega)]^2}}{\|(\vec{f}_n, g_n)\|_{[\mathrm{H}^{-1}(\Omega)]^3}} \right).$$

is a metric function on $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ and H-convergence is equivalent to convergence with respect to d.



Theorem (Irrelevance of boundary conditions)

Let $(\mathbf{P}_n, S_n) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *be a sequence that H-converges to* $(\mathbf{P}^*, S^*)$*. For any two sequences* $\vec{y}_n$ *and* z_n *such that*

$$\begin{aligned} -\mathrm{div}\,(\mathbf{P}_n \nabla \vec{y}_n) + S_n(\vec{y}_n + z_n) = \vec{f}_n &\longrightarrow \vec{f} \quad \text{in } [\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega)]^2, \\ -\mathrm{div}\,(S_n(\vec{y}_n + z_n)) = g_n &\longrightarrow g \quad \text{in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega), \\ \vec{y}_n &\longrightarrow \vec{y} \quad \text{in } [\mathrm{H}^{1}_{\mathrm{loc}}(\Omega)]^2, \\ z_n &\longrightarrow z \quad \text{in } \mathrm{H}^{1}_{\mathrm{loc}}(\Omega). \end{aligned}$$

Then the weak convergences $\mathbf{P}_n \nabla \vec{y}_n \rightharpoonup \mathbf{P}^* \nabla \vec{y}$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{M}_2)$ *and* $S_n(\vec{y}_n + z_n) \rightharpoonup S^*(\vec{y} + z)$ *in* $[\mathrm{L}^2_{\mathrm{loc}}(\Omega)]^2$ *hold.*



Theorem (Locality of H-convergence)

Let $\mathbf{P}_n$ and $\mathbf{Q}_n$ be sequences in $\mathcal{T}(\Omega, \alpha_1, \beta_1)$, and let S_n and T_n be sequences in $\mathcal{M}(\Omega, \alpha_2, \beta_2)$ be such that $(\mathbf{P}_n, S_n)$ H-converges to $(\mathbf{P}^*, S^*)$ and $(\mathbf{Q}_n, T_n)$ H-converges to $(\mathbf{Q}^*, T^*)$. Let ω be a compactly embedded in Ω. If $(\mathbf{P}_n, S_n) = (\mathbf{Q}_n, T_n)$ in ω, then $(\mathbf{P}^*, S^*) = (\mathbf{Q}^*, T^*)$ in ω.



Theorem

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *that either converges strongly to* $(\mathbf{P}^*, S^*) \in \mathrm{L}^1(\Omega, \mathcal{L}(\mathcal{M}_2, \mathcal{M}_2)) \times \mathrm{L}^1(\Omega, \mathcal{M}_2)$*, or converges to* $(\mathbf{P}^*, S^*)$ *a.e. in* Ω*. Then* $(\mathbf{P}_n, S_n)$ *H-converges to* $(\mathbf{P}^*, S^*)$*.*



Theorem

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *which H-converges to* $(\mathbf{P}^*, S^*) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$*. Assume that* $\mathbf{P}_n$ *and* S_n *are symmetric and that*

$$
\begin{aligned}
&\mathbf{P}_n \overset{*}{\rightharpoonup} \overline{\mathbf{P}} \quad \text{in } \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})),\\
&(\mathbf{P}_n)^{-1} \overset{*}{\rightharpoonup} \underline{\mathbf{P}}^{-1} \quad \text{in } \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})),\\
&S_n \overset{*}{\rightharpoonup} \overline{S} \quad \text{in } \mathrm{L}^\infty(\Omega; \mathrm{Sym}),\\
&(S_n)^{-1} \overset{*}{\rightharpoonup} \underline{S}^{-1} \quad \text{in } \mathrm{L}^\infty(\Omega; \mathrm{Sym}).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\underline{\mathbf{P}} A : A \leq \mathbf{P}^* A : A \leq \overline{\mathbf{P}} A : A, \qquad A \in \mathrm{Sym},\\
&\underline{S}\xi \cdot \xi \leq S^*\xi \cdot \xi \leq \overline{S}\xi \cdot \xi, \qquad \xi \in \mathbf{R}^2.
\end{aligned}
$$



Theorem (Energy convergence)

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *that H-converges to* $(\mathbf{P}^*, S^*)$*. For any* $(\vec{f}, g) \in [\mathrm{H}^{-1}(\Omega)]^3$*, let* $(\vec{\varphi}_n, w_n)$ *be a sequence of solutions to*

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}_n \nabla \vec{\varphi}_n) + S_n(\vec{\varphi}_n + \nabla w_n) = \vec{f} \\ -\mathrm{div}\,(S_n(\vec{\varphi}_n + \nabla w_n) = g \\ (\vec{\varphi}_n, w_n) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}.$$

The following convergences hold

$$\mathbf{P}_n \nabla \vec{\varphi}_n : \nabla \vec{\varphi}_n \longrightarrow \mathbf{P}^* \nabla \vec{\varphi} : \nabla \vec{\varphi},$$

$$S_n(\vec{\varphi}_n + \nabla w_n) \cdot (\vec{\varphi}_n + \nabla w_n) \longrightarrow S^*(\vec{\varphi} + \nabla w) \cdot (\vec{\varphi} + \nabla w)$$

in sense of distributions, and

$$\int_\Omega \mathbf{P}_n \nabla \vec{\varphi}_n : \nabla \vec{\varphi}_n + S_n(\vec{\varphi}_n + \nabla w_n) \cdot (\vec{\varphi}_n + \nabla w_n)\,\mathrm{d}\mathbf{x} \longrightarrow \int_\Omega \mathbf{P}^* \nabla \vec{\varphi} : \nabla \vec{\varphi} + S^*(\vec{\varphi} + \nabla w) \cdot (\vec{\varphi} + \nabla w)\,\mathrm{d}\mathbf{x}$$

where $(\vec{\varphi}, w)$ *denotes the solution of*

$$\begin{cases} -\mathrm{div}\,(\mathbf{P} \nabla \vec{\varphi}) + S(\vec{\varphi} + \nabla w) = \vec{f} \\ -\mathrm{div}\,(S(\vec{\varphi} + \nabla w) = g \\ (\vec{\varphi}, w) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}$$



Theorem

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$. *If* $(\mathbf{P}_n, S_n)$ *H-converges to* $(\mathbf{P}^*, S^*) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$, *then the sequence* $((\mathbf{P}_n)^T, (S_n)^T)$ *H-converges to*

$$((\mathbf{P}^*)^T, (S^*)^T) \in \mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2).$$



Theorem (Ordering property)

Let $(\mathbf{P}_n, S_n)$ *and* $(\mathbf{Q}_n, T_n)$ *symmetric sequences in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *that converge to* $(\mathbf{P}^*, S^*)$ *and* $(\mathbf{Q}^*, T^*)$, *respectively. Assume that* $\mathbf{P}_n, \mathbf{Q}_n$ *and* S_n, T_n *are ordered in sense that, for each* $n \in \mathbf{N}$

$$\mathbf{P}_n A : A \leq \mathbf{Q}_n A : A, \quad A \in \mathcal{M}_2,$$
$$S_n \xi \cdot \xi \leq T_n \xi \cdot \xi, \quad \xi \in \mathbf{R}^2.$$

Then

$$\mathbf{P}^* A : A \leq \mathbf{Q}^* A : A, \quad A \in \mathcal{M}_2,$$
$$S^* \xi \cdot \xi \leq T^* \xi \cdot \xi, \quad \xi \in \mathbf{R}^2.$$



Definition

Let $(\mathbf{P}_n, S_n)$ be a sequence in $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ which H-converges to $(\mathbf{P}^*, S^*)$. For $i, j = 1, 2$, let $\vec{\psi}_n^{ij} \in [\mathrm{H}^1(\Omega)]^2$ and $v_n^i \in \mathrm{H}^1(\Omega)$ be oscillating test functions satisfying

$$
\begin{aligned}
\vec{\psi}_n^{ij} &\longrightarrow x_i e_j \quad \text{in} \quad [\mathrm{H}^1(\Omega)]^2\,, \\
v_n^j &\longrightarrow x_j \quad \text{in} \quad \mathrm{H}^1(\Omega)\,, \\
F_n^{ij} = -\mathrm{div}\,(\mathbf{P}^n \nabla \vec{\psi}_n^{ij}) &\longrightarrow F^{ij} \quad \text{in} \quad [\mathrm{H}^{-1}(\Omega)]^2\,, \\
G_n^i = -\mathrm{div}\,(S_n \nabla v_n^i) &\longrightarrow G^i \quad \text{in} \quad \mathrm{H}^{-1}(\Omega)\,,
\end{aligned}
$$

where F^{ij} and G^i are some elements of $[\mathrm{H}^{-1}(\Omega)]^2$ and $\mathrm{H}^{-1}(\Omega)$, respectively. The pair $(\mathbf{A}_n, B_n)$ is called corrector, where $\mathbf{A}_n^{ijkl} := [\nabla \vec{\psi}_n^{ij}]_{kl}$ and $B_n^{ij} = (\nabla v_n^j)_i$.



Theorem (Uniqueness of the corrector)

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *that H-converges to* $(\mathbf{P}^*, S^*)$*. Corrector* $(\mathbf{A}_n, B_n)$ *is unique in the sense that, if there are two correctors* $(\mathbf{A}_n, B_n)$ *and* $(\widetilde{\mathbf{A}}_n, \widetilde{B}_n)$*, then*

$$(\mathbf{A}_n - \widetilde{\mathbf{A}}_n, B_n - \widetilde{B}_n) \longrightarrow (0,0) \quad \textit{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathcal{M}_2, \mathcal{M}_2)) \times \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{M}_2).$$

Lemma

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *which* H*-converges to* $(\mathbf{P}^*, S^*)$*, and let* (A_n, B_n) *be an associated corrector. Then*

$$\begin{aligned} \mathbf{A}_n &\rightharpoonup \mathbf{I}_4 && \text{in } \mathrm{L}^2(\Omega; \mathcal{L}(\mathcal{M}_2, \mathcal{M}_2)), \\ B_n &\rightharpoonup I_2 && \text{in } \mathrm{L}^2(\Omega; \mathcal{M}_2). \end{aligned}$$



Theorem

Let $(\mathbf{P}_n, S_n)$ *be a sequence in* $\mathcal{T}(\Omega, \alpha_1, \beta_1) \times \mathcal{M}(\Omega, \alpha_2, \beta_2)$ *which* H*-converges to* $(\mathbf{P}^*, S^*)$*. For* $(\vec{f}, g) \in [\mathrm{H}^{-1}(\Omega)]^3$*, let* $(\vec{\varphi}_n, w_n)$ *be a solution to*

$$\begin{cases} -\mathrm{div}\,(\mathbf{P}_n \nabla \vec{\varphi}_n) + S_n(\vec{\varphi}_n + \nabla w_n) = \vec{f} \\ -\mathrm{div}\,(S_n(\vec{\varphi}_n + \nabla w_n) = g \\ (\vec{\varphi}_n, w_n) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}.$$

Let $(\vec{\varphi}, w) \in \mathrm{H}_0^1(\Omega)$ *be the weak limit of* $(\vec{\varphi}_n, w_n)$*, that is, the solution to*

$$\begin{cases} -\mathrm{div}\,(\mathbf{P} \nabla \vec{\varphi}) + S(\vec{\varphi} + \nabla w) = \vec{f} \text{ in } \Omega \\ -\mathrm{div}\,(S(\vec{\varphi} + \nabla w) = g \text{ in } \Omega \\ (\vec{\varphi}, w) \in [\mathrm{H}_0^1(\Omega)]^3 \end{cases}.$$

Then $\nabla \vec{\varphi}_n = \mathbf{A}_n \nabla \vec{\varphi} + R_n$ *and* $\vec{\varphi}_n + \nabla w_n = B_n(\vec{\varphi} + \nabla w) + \vec{r}_n$*, where* $(\mathbf{A}_n, B_n)$ *is a corrector and* $(R_n, \vec{r}_n) \to (0, 0)$ *strongly in* $[\mathrm{L}^1_{\mathrm{loc}}(\Omega)]^3$*.*

Further research



- Periodic homogenization
- Small amplitude homogenization
- G-closure
- Optimal design

Thank you for your attention!