Homogenisation of nonlocal linear diffusion problems with fractional operators

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Motivation: Stationary diffusion equation

$\Omega \subseteq \mathbb{R}^d$ open and bounded set representing a medium

$$\begin{cases} -\operatorname{div}(\mathbf{A}\nabla u) = f, \\ u|_{\partial\Omega} = 0. \end{cases}$$

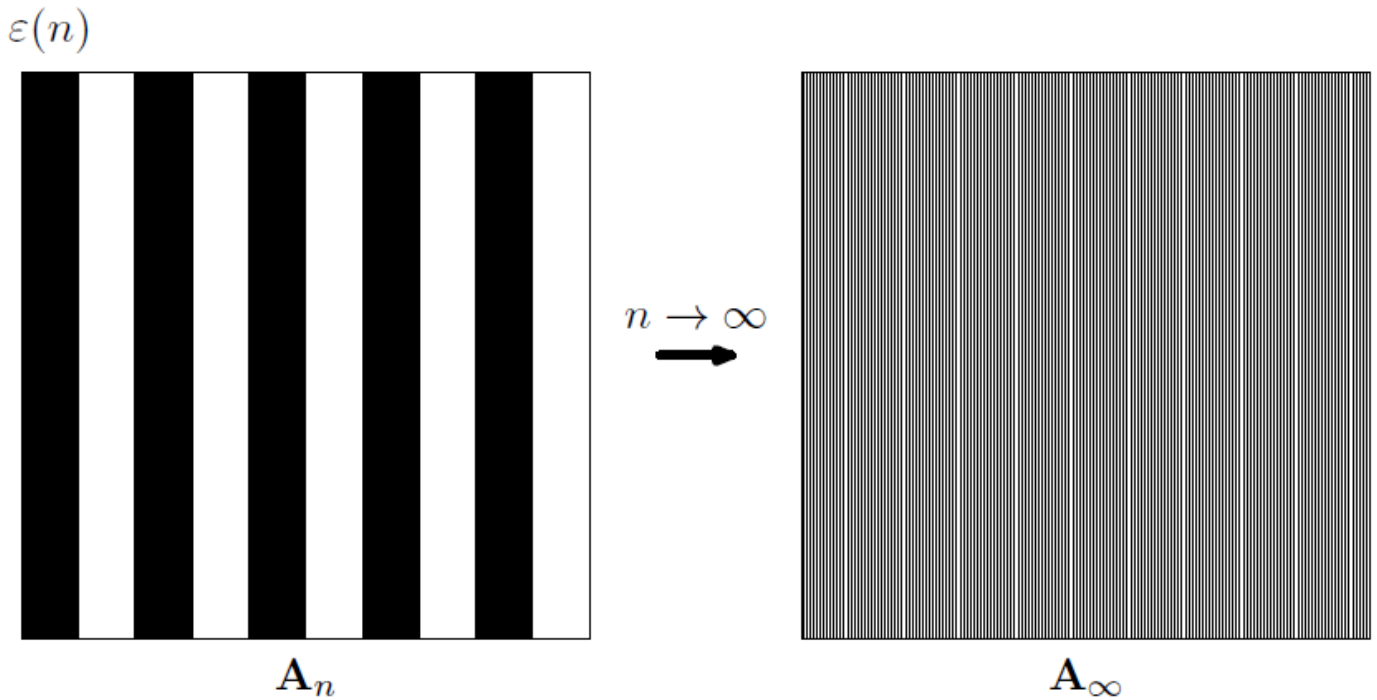
Physical interpretation:

- u – temperature / electric potential
- $\mathbf{A}$ – thermal/electrical conductivity of the material that constitutes Ω
- f – source term
- $\mathbf{A}\nabla u$ – heat / electric flux

Stationary diffusion with highly oscillatory coefficients

$\Omega \subseteq \mathbb{R}^d$ open and bounded, $f \in H^{-1}(\Omega)$

$$\begin{cases} -\operatorname{div}(\mathbf{A}_n \nabla u_n) = f, \\ u_n|_{\partial\Omega} = 0. \end{cases}$$



Homogenisation: If $\mathbf{A}_n$ oscillates highly, what is the effective coefficient $\mathbf{A}_\infty$?

H-convergence (classical; Murat–Tartar, 1970s)

For $0 < \alpha < \beta$, $\Omega \subseteq \mathbb{R}^d$ open and bounded, define the set of admissible coefficients:

$$\mathcal{M}(\alpha, \beta; \Omega) := \left\{ \mathbf{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d}) : \ \mathbf{A}(x)\xi \cdot \xi \geq \alpha|\xi|^2, \mathbf{A}^{-1}(x)\xi \cdot \xi \geq \frac{1}{\beta}|\xi|^2 \right\}.$$

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H-convergence (on $\mathcal{M}(\alpha, \beta; \Omega)$)

$(\mathbf{A}_n)$ H-converges to $\mathbf{A}$ $(\mathbf{A}_n \xrightarrow{H} \mathbf{A})$ if for every $f \in H^{-1}(\Omega)$

$$u_n \xrightarrow{H_0^1(\Omega)} u \quad \text{and} \quad \mathbf{A}_n \nabla u_n \xrightarrow{L^2(\Omega;\mathbb{R}^d)} \mathbf{A}\nabla u, \quad n \to \infty,$$

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Key properties:

- **Metrisability and compactness:** $(\mathcal{M}(\alpha, \beta; \Omega), \tau_H)$ is compact and metrisable.
- Application in optimal design problems with composite materials.

Nonlocal diffusion problem with the Riesz fractional operators

New equation: fractional divergence form

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\,\nabla^{\mathrm{s}} u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

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Riesz fractional gradient & divergence

Let $s \in (0,1)$ and $\mu_s = 2^s \pi^{-d/2} \Gamma\left(\frac{d+s+1}{2}\right) \Gamma\left(\frac{1-s}{2}\right)^{-1}$.

$$\nabla^{s}\varphi(x) = \mu_s \int_{\mathbb{R}^d} \frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|^{d+s+1}}\,dy$$

$$\operatorname{div}^{s}\psi(x) = \mu_s \int_{\mathbb{R}^d} \frac{(\psi(x)-\psi(y))\cdot(x-y)}{|x-y|^{d+s+1}}\,dy$$

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Suitable space of test functions

$$\mathcal{T}(\mathbb{R}^d) = \{\varphi \in \mathrm{C}^\infty(\mathbb{R}^d) : \partial^\alpha \varphi \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{C}_0(\mathbb{R}^d), \text{ for every } \alpha \in \mathbb{N}_0^d\},$$

endowed with the topology given by the countable family of norms $|\cdot|_k$, $k \in \mathbb{N}_0$,

$$|\varphi|_k = \max\left\{ \|\partial^\alpha \varphi\|_{\mathrm{L}^1(\mathbb{R}^d)}, \|\partial^\alpha \varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)} : \alpha \in \mathbb{N}_0^d,\ 0 \le |\alpha| \le k \right\}.$$

We have $\mathcal{D}(\mathbb{R}^d) \overset{\text{dense}}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d) \overset{\text{dense}}{\hookrightarrow} \mathcal{T}(\mathbb{R}^d)$ and $\mathcal{T}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d)$.

Properties of Fractional Differential Operators

- **Continuity ans canonicity:** ∇^{s} (resp. $\mathrm{div}^{\mathrm{s}}$) is unique (up to the scalar) translationally and rotationally invariant, and s-homogeneous continuous operator $\mathcal{T} \to \mathcal{T}^d$ (resp. $\mathcal{T}^d \to \mathcal{T}$).

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- **Representation via Riesz potential:** For $\varphi \in \mathcal{T}$ and $\psi \in \mathcal{T}^d$, it holds

$$\nabla^{\mathrm{s}}\varphi = \nabla I_{1-s}\varphi = I_{1-s}\nabla\varphi \quad \text{and} \quad \mathrm{div}^{\mathrm{s}}\,\psi = \mathrm{div}\, I_{1-s}\psi = I_{1-s}\,\mathrm{div}\,\psi,$$

where for $\alpha \in (0,d)$ and a measurable function $f : \mathbb{R}^d \to \mathbb{R}$, the Riesz potential is given by

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$${}_{\mathcal{T}'^d}\langle \nabla^{\mathrm{s}} F, f\rangle_{\mathcal{T}^d} = -{}_{\mathcal{T}'}\langle F, \operatorname{div}^{\mathrm{s}} f\rangle_{\mathcal{T}} \quad \text{and} \quad {}_{\mathcal{T}'}\langle \operatorname{div}^{\mathrm{s}} F, f\rangle_{\mathcal{T}} = -{}_{\mathcal{T}'^d}\langle F, \nabla^{\mathrm{s}} f\rangle_{\mathcal{T}^d}$$

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- **Fourier transform:** $\widehat{\nabla^s f}(\xi) = i\xi|\xi|^{s-1}\widehat{f}(\xi)$

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M. Šilhavý. *Fractional vector analysis based on invariance requirements (critique of coordinate approaches)*, Continuum Mech. Thermodyn. **32** (2019) 207–228.


T. Shieh and D. Spector. *On a new class of fractional partial differential equations*, Adv. Calc. Var **8** (2015) 321–336.

Fractional Sobolev Spaces

For $s \in (0,1)$, define the s-norm: $\|u\|_s^2 := \|u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla^s u\|^2_{L^2(\mathbb{R}^d)^d}$, and then

$$H^s(\mathbb{R}^d) := \text{ completion of the space } (C_c^\infty(\mathbb{R}^d), \|\cdot\|_s), \qquad H^{-s}(\mathbb{R}^d) := \big(H^s(\mathbb{R}^d)\big)'.$$

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For open and bounded $\Omega \subseteq \mathbb{R}^d$:

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Essential tools (valid for $d \geq 1$):

1. $I_{1-s} : H_0^s(\Omega) \to H^1_{\text{loc}(\mathbb{R}^d)}$ is linear, continuous and $\nabla(I_{1-s}u) = \nabla^s u$ a.e. in Ω.
2. **Poincaré inequality:** $\|u\|_{L^2(\Omega)} \lesssim \|\nabla^s u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$, $u \in H_0^s(\Omega)$.
3. **Rellich–Kondrašov:** $H_0^s(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$.
4. **Compactness on the complement**: for $\Omega \Subset \Omega_1$, the mapping $u \mapsto \nabla^s u|_{\Omega_1^c}$ is *compact*, from $H_0^s(\Omega)$ to $L^2(\Omega_1^c;\mathbb{R}^d)$.[1]

[1] C. Kreisbeck and H. Schönberger. *Quasiconvexity in the fractional calculus of variations: characterization of lower semicontinuity and relaxation*, Nonlinear Analysis **215** (2022).

Back to the fractional diffusion problem

Fractional Diffusion: Weak Formulation & Well-Posedness

For $\mathbf{A} \in \mathcal{M}(\alpha, \beta; \mathbb{R}^d)$ and $f \in H^{-s}(\Omega)$, consider

$$\begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\,\nabla^{\mathrm{s}} u) = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^d \setminus \Omega, \end{cases} \qquad \text{or} \qquad \begin{cases} -\operatorname{div}^{\mathrm{s}}(\mathbf{A}\,\nabla^{\mathrm{s}} u) = f, \\ u \in H^s_0(\Omega). \end{cases}$$

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Weak formulation: find $u \in H^s_0(\Omega)$ such that

$$\int_{\mathbb{R}^d} \mathbf{A}(x)\ \nabla^{\mathrm{s}} u(x) \cdot \nabla^{\mathrm{s}} v(x)\, dx = {}_{H^{-s}(\Omega)}\langle f, v\rangle_{H^s_0(\Omega)}\,, \quad \text{for every } v \in H^s_0(\Omega)\,.$$

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[CCM] M. Caponi, A. Carbotti, and A. Maione. *H-compactness for nonlocal linear operators in fractional divergence form*, Calc. Var. Partial Differential Equations, **64**.290 (2025).

Well-posedness [CCM; extended to $d \geq 1$]

For every $f \in H^{-s}(\Omega)$ there exists a unique weak solution $u \in H^s_0(\Omega)$, satisfying

$$\|\nabla^{\mathrm{s}} u\|_{L^2(\mathbb{R}^d)^d} \leq \tfrac{1}{\alpha}\|f\|_{H^{-s}(\Omega)}.$$

Note: Coefficients $\mathbf{A}$ must be defined on *all of* $\mathbb{R}^d$ due to the nonlocality of ∇^{s}.

H^s-convergence (on $\mathcal{M}(\alpha, \beta; \mathbb{R}^d)$)

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For some $\mathbf{A}_0 \in \mathcal{M}(\alpha,\beta;\mathbb{R}^d)$, define

$$\mathcal{M}(\alpha,\beta;\Omega,\mathbf{A}_0) := \{\mathbf{A} \in \mathcal{M}(\alpha,\beta;\mathbb{R}^d) : \mathbf{A}|_{\mathbb{R}^d\setminus\Omega} = \mathbf{A}_0|_{\mathbb{R}^d\setminus\Omega}\}\,.$$

[CCM] H^s-compactness and H^s-convergence on $\mathcal{M}(\alpha,\beta;\Omega,\mathbf{A}_0)$

Let $(\mathbf{A}_n) \subset \mathcal{M}(\alpha,\beta;\Omega,\mathbf{A}_0)$, $s \in (0,1)$ and $d \geq 2$.

- **Compactness:** There exist a subsequence $\mathbf{A}_{n_k}$ and a limit matrix $\mathbf{A} \in \mathcal{M}(\alpha,\beta;\Omega,\mathbf{A}_0)$ such that $\mathbf{A}_{n_k} \xrightarrow{H^s} \mathbf{A}$. Furthermore, the H^s-limit is unique.
- **Characterisation via H-convergence:** $\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \iff \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega$

H^s-convergence: our contribution



[BBCEJW] A. Buchinger, K. Burazin, I. Crnjac, M. Erceg, M. Jolić, M. Waurick. *Characterisation of homogenisation for nonlocal diffusion by local topologies*, https://arxiv.org/abs/2601.17579 (2026).

Characterisation of H^s-convergence on $\mathcal{M}(\alpha, \beta; \mathbb{R}^d)$, $s \in (0,1)$, $d \geq 1$:

$\Omega \subseteq \mathbb{R}^d$ open, bounded and Lebesgue measure of its boundary is zero

$$\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \iff \mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega \text{ and } \mathbf{A}_n|_{\mathbb{R}^d \setminus \mathrm{Cl}\,\Omega} \xrightarrow{*} \mathbf{A}|_{\mathbb{R}^d \setminus \mathrm{Cl}\,\Omega}$$

H^s-convergence: our contribution



[BBCEJW] A. Buchinger, K. Burazin, I. Crnjac, M. Erceg, M. Jolić, M. Waurick. *Characterisation of homogenisation for nonlocal diffusion by local topologies*, https://arxiv.org/abs/2601.17579 (2026).

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Properties of H^s-convergence:

- Compactness and metrisability $\quad d_{H^s}(\mathbf{A}, \mathbf{B}) = d_H(\mathbf{A}|_\Omega, \mathbf{B}|_\Omega) + d_*(\mathbf{A}|_{\mathbb{R}^d \setminus \operatorname{Cl}\Omega}, \mathbf{B}|_{\mathbb{R}^d \setminus \operatorname{Cl}\Omega})$
- H^s-convergence is *local*

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- H^s-convergence is *local*
- One-dimensional case:
  The H^s-limit is the *harmonic mean* on Ω and the *arithmetic mean* on the complement.

  Note: In the classical 1-dim case ($s = 1$) we have strong convergence of the fluxes $\mathbf{A}_n u_n'$, which no longer holds in the fractional ($s \in (0,1)$) case for $\mathbf{A}_n \, \mathrm{D}^s u_n$. This is the consequence of the property:

  the operator $\mathrm{D}^s : L^2(\mathbb{R}) \to H^{-s}(\Omega)$ has an infinite-dimensional kernel.
- H^s-converges fits the general theory of nonlocal H-convergence (Waurick 2018/2025)

Key ideas of the proof

1. Relations between ∇ and ∇^{s}:

$$\nabla^{\mathrm{s}} u = \nabla(I_{1-s}u), \text{ for } u \in H_0^s(\Omega) \quad \text{and} \quad \nabla w = \nabla^{\mathrm{s}}\big((-\Delta)^{\frac{1-s}{2}} w\big), \text{ for } w \in H^1(\mathbb{R}^d)$$

2. Poincaré inequality: $\|u\|_{L^2(\Omega)} \lesssim \|\nabla^{\mathrm{s}} u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$, $u \in H_0^s(\Omega)$.

3. Rellich–Kondrašov: $u_n \xrightarrow{H_0^s(\Omega)} u \implies u_n \xrightarrow{L^2(\Omega)} u.$

4. Compactness on the complement:

$$\text{for } \Omega \Subset \Omega_1\text{:} \quad u_n \xrightarrow{H_0^s(\Omega)} u \implies \nabla^{\mathrm{s}} u_n|_{\Omega_1^c} \xrightarrow{L^2(\Omega_1^c)} \nabla^{\mathrm{s}} u|_{\Omega_1^c}$$

5. **Fractional compensated compactness:**

$$v_n \xrightarrow{H_0^s(\Omega)} v, \quad u_n \xrightarrow{L^2(\mathbb{R}^d)} u, \quad \operatorname{div}^{\mathrm{s}} u_n \xrightarrow{\mathrm{H}^{-s}(\Omega)} \operatorname{div}^{\mathrm{s}} u \implies u_n \cdot \nabla^{\mathrm{s}} v_n \xrightarrow{\mathcal{D}'(\Omega)} u \cdot \nabla^{\mathrm{s}} v$$

Key ideas of the proof

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Idea of the proof.

- The analysis of $\mathbf{A}_n \nabla^{\mathrm{s}} u_n$ on Ω uses that $\mathbf{A}_n|_\Omega \xrightarrow{H} \mathbf{A}|_\Omega$, and hinges on (1) (enabling the use of classical and fractional comp. compactness). The same idea is present in [CCM].
- The convergence of $\mathbf{A}_n \nabla^{\mathrm{s}} u_n$ on Ω^c is obtained due to (4).

Nonlocal operators with finite domain of interaction

Finite-horizon fractional operators

For an open, bounded set $\Omega \subseteq \mathbb{R}^d$ and fixed $\delta > 0$, define δ-closure of Ω

$$\Omega_\delta := \Omega + B(0,\delta) = \{x \in \mathbb{R}^d : d(x,\Omega) < \delta\},$$

and a smooth, non-negative, radially decreasing cut-off function $w_\delta : \mathbb{R}^d \to [0,\infty)$ with $\operatorname{supp} w_\delta \subseteq B(0,\delta)$.

δ-horizon fractional gradient, $s \in (0,1)$

$$\mathrm{D}^s_\delta \varphi(x) = \mu_s \int_{\mathbb{R}^d} \frac{(\varphi(x) - \varphi(y))(x-y)}{|x-y|^{d+s+1}} w_\delta(x-y)\, dy$$



J. C. Bellido, J. Cueto, and C. Mora-Corral. *Non-local gradients in bounded domains motivated by continuum mechanics: Fundamental theorem of calculus and embeddings.* Advances in Nonlinear Analysis 12.1 (2023)



J. Cueto, C. Kreisbeck, and H. Schönberger. *A variational theory for integral functionals involving finite-horizon fractional gradients.* Fractional Calculus and Applied Analysis 26 (2023)

- $\mathrm{D}^s_\delta : \mathcal{S} \to \mathcal{S}$
- δ-fractional Sobolev spaces: $\|u\|^2_{s,\delta} = \|u\|^2_{L^2} + \|\mathrm{D}^s_\delta u\|_{L^2}$, $H^{s,\delta}_0(\Omega) \equiv H^s_0(\Omega)$
- Poincare, Rellich-Kondrašov
- compactness on the complement
- δ-fractional div-rot lemma

Work in progress

δ-fractional homogenisation problem

$$\begin{cases} -\mathrm{div}^s_\delta(\mathbf{A}_n \, \mathrm{D}^s_\delta u_n) = f, \\ u \in H^{s,\delta}_0(\Omega). \end{cases}$$

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Asymptotic and Comparison

- when $\delta \to \infty$: $\mathrm{D}^s_\delta \to \nabla^s$, as a linear operator from $H^s(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$
- $\mathbf{A}_n \xrightarrow{H^s} \mathbf{A} \iff \forall \delta > 0 \quad \mathbf{A}_n|_{\Omega_\delta} \xrightarrow{H^{s,\delta}} \mathbf{A}|_{\Omega_\delta}$
- Finite-domain approximation: The solution and the associated flux of the homogenised H^s-problem can be approximated by a sequence of the solutions and fluxes of δ-horizon fractional problems.

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Optimal design

Thank you for your attention!